Optimal control of a 1D diffusion process with a team of mobile actuators under jointly optimal guidance

Sheng Cheng and Derek A. Paley

***Abstract*— This paper describes an optimization framework to control a distributed parameter system (DPS) using a team of mobile actuators. The optimization simultaneously seeks efficient guidance of the mobile actuators and effective control of the DPS such that an integrated cost function associated with both the mobile actuators and the DPS is minimized. Since the optimization does not have a constraint restricting the actuators to the domain of the DPS, the actuators may actuate outside the domain with no contribution towards regulating the DPS. We show that, under certain conditions, any guidance that steers the mobile actuators out of the spatial domain is non-optimal. This result implies that optimal guidance is guaranteed to restrict the actuators to the domain even without explicit constraints. A gradient-descent method solves the integrated optimization problem numerically using its finite-dimensional approximation. We also synthesize the optimal feedback control of the DPS given jointly optimal guidance of the mobile actuators. A numerical example illustrates the optimization framework and the solution method.**

I. Introduction

Recent development of mobile robots (unmanned aerial vehicles, terrestrial robots, and underwater vehicles) has greatly extended the type of distributed parameter system (DPS) over which mobile actuation and sensing can be deployed. Such a system is often modeled by a partial differential equation (PDE), which varies in both time and space. Exemplary applications of mobile control and estimation of a DPS can be found in a thermal manufacturing process [1], monitoring and neutralizing groundwater contamination [2], and wildfire monitoring [3].

We propose an optimization framework that simultaneously solves for the guidance of a team of mobile actuators and the control of a DPS. We consider a 1D diffusion process as the DPS for the convenience of the state-space representation and explicit expression of the optimal feedback control. The framework minimizes an integrated cost function, evaluating both actuator motions and DPS regulation, subject to the dynamics of the mobile actuators and the DPS. The integrated problem can better address the mobile actuator and the DPS as a unified system, instead of solely regulating the DPS. Furthermore, the additional degree of freedom endowed by mobility yields improved control performance in comparison to using fixed actuators.

Sheng Cheng is a graduate student in the Department of Electrical and Computer Engineering and the Institute for Systems Research at the University of Maryland, College Park, MD 20742, USA. (email:{cheng@terpmail.umd.edu})

Derek A. Paley is the Willis H. Young Jr. Professor in the Department of Aerospace Engineering and the Institute for Systems Research at the University of Maryland, College Park, MD 20742, USA. (email:{dpaley@umd.edu})

The proposed framework is well-suited for the limited onboard resources of mobile actuators in the following two aspects: (1) it adopts a finite-horizon optimization scheme that characterizes the resource limitation more precisely than the approaches that do not specify a terminal time such as an infinite-horizon optimization or Lyapunov-based method and (2) it provides an intermediate step for the optimization problem that characterizes the limited resources as inequality constraints, because the constraints can be used to augment the cost function and turned into the proposed form using the method of Lagrange multipliers.

The control of a DPS can be categorized by the location of the actuation. The same category applies to estimation and parameter identification by sensors, but the literature review here focus on control and actuation for brevity. When actuation occurs on the boundary of the spatial domain, it is called boundary control. Representative work in boundary control is mainly conducted by Krstic and collaborators [4]. When actuation acts in the interior of the spatial domain, it is called distributed control. For distributed control, the DPS is actuated by in-domain actuators that are either fixed or movable. For fixed actuators, the problem of determining the location of actuators is called the actuator placement problem. Actuator placement has been studied for optimality in the sense of linear-quadratic (LQ) [5], $\mathbb{H}_2$ [6], and maximum controllability [7]. For mobile actuators, guidance of the actuators is designed to improve the control performance in comparison to fixed actuators. Various criteria have been proposed for guidance, for example, using LQ control [1], [8], [9], Lyapunov-based methods [10], [11], constant velocity [12], and Centroidal Voronoi Tessellation [13], [14].

Among the existing approaches, only [9] considers an integrated cost of actuator motion and DPS regulation. However, the numerical method proposed in [9] reduces the admissible set of a mobile actuator from the entire spatial domain to a set of pre-specified points, and this is suboptimal with respect to the original admissible set. Also, the discretization induces extra constraints on the displacement of actuator locations between consecutive decision times to mitigate the violation of the continuous dynamics of the mobile actuator.

The contributions of this paper are summarized as follows: (1) the problem of regulating a diffusion process with a *team* of mobile actuators is formulated as an integrated optimization problem; (2) conditions are showed under which optimal guidance is guaranteed to restrict the mobile actuators to the spatial domain even without explicit constraints; and (3) a gradient-descent method is applied to numerically solve the approximation of the formulated problem for the jointly optimal guidance of the mobile actuators and the (feedback) control of the DPS. The proposed framework provides a new approach to simultaneously design the guidance and actuation of a team of mobile actuators to control a DPS. Potential applications include wildland firefighting using unmanned aerial vehicles and oil spill removal using autonomous skimmer boats.

A. Notation and terminology

The paper adopts the following notation. The symbol $\mathbb{R}$ denotes the set of real numbers. The interior and boundary of a set M are denoted by $\mathrm{int}(M)$ and ∂M, respectively. The n-nary Cartesian power of a set M is denoted by M^n. Calligraphic letters represent operators and abstract spaces. The direct sum of vector spaces V_1 and V_2 is denoted by $V_1 \oplus V_2$. An embedding is denoted by $\hookrightarrow$. We use $|\cdot|$ and $\|\cdot\|$ for the absolute value and Euclidean norm, respectively, with no subscript attached. The superscript * denotes an optimal variable or an optimal value, whereas * denotes the adjoint of a linear operator. The transpose of a matrix A is denoted by A^T, whereas an $n \times n$-dimensional identity matrix is denoted by I_n. An $n \times n$-dimensional diagonal matrix with elements of vector $[a_1, a_2, \dots, a_n]$ on the main diagonal is denoted by $\mathrm{diag}(a_1, a_2, \dots, a_n)$. The derivative of a function f evaluated at x is denoted by $f'(x)$. The term *guidance* refers to the control of the mobile actuators, whereas the term *control* refers to the control of (or, equivalently, the actuation input to) the DPS.

B. Paper organization

Section II introduces relevant mathematical background, including representation of a PDE by an infinite-dimensional system, the associated LQ optimal control, and its finite-dimensional approximation. Section III formulates the integrated optimization problem, states a theorem of optimal guidance on restricting the mobile actuators to the spatial domain without explicit constraints, and synthesizes a feedback control of the DPS. Section IV details a solution method that applies a gradient-descent method and the Galerkin approximation scheme to solve the resulting finite-dimensional integrated problem. A numerical example is provided to illustrate optimal guidance and control solved by the proposed method. Section V summarizes the paper and discusses ongoing work.

II. Background

Consider controlling a 1D diffusion process with m mobile actuators modeled by the following PDE:

$$\frac{\partial z(x,t)}{\partial t} = a\frac{\partial^2 z(x,t)}{\partial x^2} + \sum_{i=1}^{m} b_i u_i(t)\delta_{\xi_i}(t), \tag{1}$$

where $z(\cdot,\cdot)$ denotes a 1D diffusion process that has a spatial component $x \in \Omega \subset \mathbb{R}$ and a time component $t \in [0, t_f]$ for a given terminal time t_f; $u(t) \in U \subset \mathbb{R}^m$ is a (vector) function that denotes the magnitude of the actuation input and is piecewise continuous in t; $\delta_x(\cdot)$ is a Dirac delta function with a unit impulse at $x(\cdot) \in \mathbb{R}$; and $\xi(\cdot) \in \mathbb{R}^m$ is a vector of the locations of the mobile actuators. We require δ_x to be zero if $x \notin \mathrm{int}(\Omega)$. The coefficients $a \in \mathbb{R}$ and $b \in \mathbb{R}^m$ denote the diffusion coefficient and input gains of the actuators, respectively. The diffusion process has a known initial condition $z(\cdot, 0) = z_0(\cdot)$ and a Dirichlet boundary condition $z(\cdot,\cdot)|_{\partial\Omega} = 0$. The following linear dynamics describe the motion of the mobile actuators:

$$\dot{\xi}(t) = \alpha\xi(t) + \beta p(t), \tag{2}$$

$$\xi(0) = \xi_0, \tag{3}$$

where

$$\alpha = \mathrm{diag}(\alpha_1, \alpha_2, \dots, \alpha_m), \qquad \beta = \mathrm{diag}(\beta_1, \beta_2, \dots, \beta_m),$$

$\alpha_i, \beta_i \in \mathbb{R}$ for $i = 1, 2, \dots, m$, and the initial locations $\xi_0 \in \mathbb{R}^m$ are given. The guidance $p(\cdot) \in \mathbb{R}^m$ of the mobile actuators is a (vector) function piecewise continuous in t.

Since partial differential equations can be formulated as differential equations on an abstract linear vector space of infinite dimension [15], we can compactly represent system (1) as an infinite-dimensional linear system

$$\dot{\mathcal{Z}}(t) = \mathcal{A}\mathcal{Z}(t) + \mathcal{B}_\xi(t)u(t), \qquad \mathcal{Z}(0) = z_0, \tag{4}$$

where $\mathcal{Z}$ belongs to a Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|_{\mathcal{H}}$. Here, the variable $\mathcal{Z}(\cdot)$ is the state of the DPS and space $\mathcal{H}$ is the state space. Let $\mathcal{V}$ be a reflexive Banach space with norm $\|\cdot\|_{\mathcal{V}}$. Let $\mathcal{V}^\star$ be the conjugate dual of $\mathcal{V}$ with $\|\cdot\|_{\mathcal{V}^\star}$ denoting the usual uniform operator norm on $\mathcal{V}^\star$. Note that $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^\star$ and both embeddings are dense and continuous [10]. In this paper, we adopt $\mathcal{H} = L^2(\Omega)$ and Sobolev spaces $\mathcal{V} = H_0^1(\Omega)$ and $\mathcal{V}^\star = H^{-1}(\Omega)$. The operator $\mathcal{A}$ is defined as $\mathcal{A}\psi = a\partial^2\psi(x)/\partial x^2$ with $\psi \in \mathrm{Dom}(\mathcal{A}) = \{\psi \in H_0^1(\Omega),\ \nabla^2\psi \in L^2(\Omega)\} = H^2(\Omega) \cap H_0^1(\Omega)$ [10]. The operator $\mathcal{B}_\xi(\cdot) \in C([0, t_f]; \mathcal{L}(U; \mathcal{V}^\star))$ is the Dirac delta in the sense of a linear operator in $\mathcal{V}^\star$ such that $\mathcal{B}_\xi(\cdot) = [b_1\delta_{\xi_1}(\cdot), b_2\delta_{\xi_2}(\cdot), \dots, b_m\delta_{\xi_m}(\cdot)]$.

Remark 1: Since $\mathcal{B}_\xi(\cdot)u$ is continuous for all $u \in U$, by [16], the evolution equation (4) has a unique mild solution. Here, a mild solution of (4) is given by

$$\mathcal{Z}(t) = F(t)\mathcal{Z}(0) + \int_0^t F(t-\tau)\mathcal{B}_\xi(\tau)u(\tau)\mathrm{d}\tau$$

if $\mathcal{Z}$ belongs to $C([0, t_f]; \mathcal{H})$, where F is a strongly continuous semigroup generated by $\mathcal{A}$.

Assuming state $\mathcal{Z}$ is available for full-state feedback control, we do not specify an output equation. Ongoing work investigates the design of a state observer to estimate the state.

Similar to a finite-dimensional system, we can formulate a linear-quadratic regulator (LQR) with the differential equation (4). A general LQR minimizes the following cost:

$$J(\mathcal{Z}, u) = \frac{1}{2}\int_0^{t_f} \Big(\langle\mathcal{Z}(t), \mathcal{Q}\mathcal{Z}(t)\rangle + u(t)^T R u(t)\Big)\mathrm{d}t + \frac{1}{2}\langle\mathcal{Z}(t_f), \mathcal{Q}_f\mathcal{Z}(t_f)\rangle, \tag{5}$$

where $\mathcal{Q} \in \mathcal{L}(\mathcal{V})$ and $\mathcal{Q}_f \in \mathcal{L}(\mathcal{V})$ are self-adjoint, nonnegative, Hilbert-Schmidt operators that evaluate the running cost and terminal cost, respectively, of the state $\mathcal{Z}$. The coefficient R is an $m \times m$-dimensional symmetric and positive-definite real matrix that weights the control effort of the DPS. By [16, Theorem 7.3], an optimal feedback control associated with a given trajectory $\xi(\cdot)$ of actuators is

$$u^*(t) = -R^{-1}\mathcal{B}_\xi^\star(t)\mathcal{S}(t)\mathcal{Z}(t), \tag{6}$$

where $\mathcal{S} : \mathcal{V} \to \mathcal{V}$ is a self-adjoint and nonnegative operator that satisfies the operator differential Riccati equation

$$\begin{aligned}\dot{\mathcal{S}}(t) = &- \mathcal{A}^\star\mathcal{S}(t) - \mathcal{S}(t)\mathcal{A} - \mathcal{Q}\\ &+ \mathcal{S}(t)\mathcal{B}_\xi(t)R^{-1}\mathcal{B}_\xi^\star(t)\mathcal{S}(t), \end{aligned} \tag{7}$$

$$\mathcal{S}(t_f) = \mathcal{Q}_f. \tag{8}$$

Remark 2: The existence of a unique mild solution to (7) can be established according to [16, Theorem 7.2]. It is omitted for space constraints.

Approximations to (4) and (7) permit numerical computation. Consider a finite-dimensional subspace $\mathcal{H}_N \subset \mathcal{H}$ with dimension N. The inner product and norm of $\mathcal{H}_N$ are inherited from that of $\mathcal{H}$. Let $P_N : \mathcal{H} \to \mathcal{H}_N$ denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_N$. We make a standard assumption [17] that

$$\lim_{N\to\infty} \|P_N\phi - \phi\|_{\mathcal{V}} = 0, \qquad \forall \phi \in \mathcal{V}. \tag{9}$$

Let $z_N(\cdot) \in \mathbb{R}^N$ and $S_N(\cdot) \in \mathbb{R}^{N\times N}$ denote the finite-dimensional approximation of $\mathcal{Z}(\cdot)$ and $\mathcal{S}(\cdot)$, respectively, where $z_N(\cdot) = P_N\mathcal{Z}(\cdot)$ and $S_N(\cdot) = P_N\mathcal{S}(\cdot)P_N$. A finite-dimensional approximation of (4) is

$$\dot{z}_N(t) = A_N z_N(t) + B_{\xi,N}(t)u(t), \tag{10}$$

$$z_N(0) = z_{N0} = P_N z_0, \tag{11}$$

where $A_N \in \mathcal{L}(\mathcal{H}_N)$ and $B_{\xi,N}(\cdot) \in \mathcal{L}(U, \mathcal{H}_N)$ are approximations of $\mathcal{A}$ and $\mathcal{B}_\xi(\cdot)$, respectively. Correspondingly, the finite-dimensional approximation of (7) is

$$\begin{aligned}\dot{S}_N(t) = &- (A_N)^T S_N(t) - S_N(t)A_N - Q_N\\ &+ S_N(t)B_{\xi,N}(t)R^{-1}B_{\xi,N}^T(t)S_N(t), \end{aligned} \tag{12}$$

$$S_N(t_f) = Q_{fN}, \tag{13}$$

where $Q_N = P_N\mathcal{Q}P_N$ and $Q_{fN} = P_N\mathcal{Q}_f P_N$. By [17, Theorem 4.1], the approximation S_N converges to $\mathcal{S}$, that is $\lim_{N\to\infty} \|S_N(t) - \mathcal{S}(t)\|_{\boldsymbol{H}} = 0$, with the convergence uniform in t for $t \in [0, t_f]$ and $\boldsymbol{H}$ denotes the Hilbert space of Hilbert-Schmidt operators from $\mathcal{H}$ to $\mathcal{H}$.

III. Problem formulation

A. Integrated optimal control problem

This subsection introduces the formulation of the integrated optimal control problem, which simultaneously solves for optimal (open-loop) guidance of the mobile actuators and optimal (open-loop) control of the DPS. Specifically, the cost functions evaluating actuator motions and DPS regulation are integrated into one cost function. Consequently, the dynamics of the DPS and that of the mobile actuator are both constraints. The integrated problem is

$$\begin{aligned}&\underset{u\in U, p\in\mathbb{R}^m}{\text{minimize}} && J(\mathcal{Z}, u) + J_{\text{m}}(\xi, p)\\ &\text{subject to} && \dot{\mathcal{Z}}(t) = \mathcal{A}\mathcal{Z}(t) + \mathcal{B}_\xi(t)u(t),\ \mathcal{Z}(0) = \mathcal{Z}_0,\\ &&& \dot{\xi}(t) = \alpha\xi(t) + \beta p(t),\ \xi(0) = \xi_0,\end{aligned} \tag{P}$$

where $J_{\text{m}}(\xi, p)$ is the cost associated with mobility, i.e.,

$$\begin{aligned}J_{\text{m}}(\xi, p) = &\frac{1}{2}\int_0^{t_f} \Big(\xi(t)^T\kappa\xi(t) + p(t)^T\gamma p(t)\Big)\mathrm{d}t\\ &+ \frac{1}{2}(\xi(t_f) - \xi_f)^T\kappa_f(\xi(t_f) - \xi_f);\end{aligned}$$

κ and κ_f are $m \times m$-dimensional symmetric and positive-semidefinite matrices, respectively; γ is an $m \times m$-dimensional positive-definite matrix; and $\xi_f \in \mathbb{R}^m$ is the vector of terminal locations for the mobile actuators. Such terminal locations may represent user-specified evacuation locations for the mobile actuators in some applications.

Problem (P) does not constrain the actuators to the spatial domain Ω. Hence, the actuators may wander out of the domain and dispense no actuation input to the DPS. The following theorem specifies a special case of (P) where mobile actuators under optimal guidance will not wander out of the domain.

Theorem 1: Consider problem (P) where the dynamics of each mobile actuator is a single integrator (i.e., $\alpha = 0$), the running state cost for the mobile actuators is zero (i.e., $\kappa = 0$), and all actuators have initial locations in the interior of the domain (i.e., $\xi_0 \in \text{int}(\Omega^m)$). If either the terminal locations are in the domain (i.e., $\xi_f \in \Omega^m$), or there is no terminal cost for the mobile actuators (i.e., $\kappa_f = 0$), then any guidance that steers the mobile actuators out of the domain Ω is non-optimal.

Remark 3: A direct consequence of Theorem 1 is that, under the conditions of Theorem 1, optimal guidance will restrict the mobile actuators to the spatial domain Ω even without explicit constraints. The proof of Theorem 1 is omitted for space constraints.

B. Synthesis of optimal feedback control of the DPS

For now, suppose we can solve problem (P) for jointly optimal control $u^*(\cdot)$ and guidance $p^*(\cdot)$, which are both in open-loop form. We will introduce a numerical approach to solve (P) using a finite-dimensional approximation of the PDE in Section IV-A. We now demonstrate how to utilize optimal guidance $p^*(\cdot)$ for the synthesis of an optimal feedback control law of the DPS.

Given the dynamics (2) and optimal guidance $p^*(\cdot)$, a unique trajectory $\xi^*(\cdot)$ of the mobile actuators is obtained to yield the control input operator $\mathcal{B}_{\xi^*}(\cdot)$. Furthermore, we can formulate a subproblem of (P) that only minimizes the cost of regulating the DPS as follows:

$$\begin{aligned}&\underset{\bar{u}(\cdot)\in U}{\text{minimize}} && J(\mathcal{Z}, \bar{u})\\ &\text{subject to} && \dot{\mathcal{Z}}(t) = \mathcal{A}\mathcal{Z}(t) + \mathcal{B}_{\xi^*}(t)\bar{u}(t),\ \mathcal{Z}(0) = z_0.\end{aligned} \tag{P1}$$

By [16, Theorem 7.2], an optimal feedback control $\bar{u}^*$ is

$$\bar{u}^*(t) = -R^{-1}\mathcal{B}^{\star}_{\xi^*}(t)\mathcal{S}(t)\mathcal{Z}^*(t), \tag{14}$$

where $\mathcal{S}(\cdot)$ is the unique mild solution of the differential Riccati equation (7) associated with an optimal trajectory ξ^*.

Remark 4: Note that the feedback control $\bar{u}^*$ and optimal guidance p^* are jointly optimal for (P). From now on, the notation $\bar{u}^*$ refers to the optimal *feedback* control associated with the jointly optimal guidance p^*.

IV. Solution method

We propose a numerical procedure to solve the optimal control of the DPS and guidance of the mobile actuators that jointly minimize the following finite-dimensional approximation of (P):

$$\begin{aligned} \underset{u\in U, p\in\mathbb{R}^m}{\text{minimize}} \quad & J_N(z_N, u) + J_{\mathrm{m}}(\xi, p) \\ \text{subject to} \quad & \dot{z}_N(t) = A_N z_N(t) + B_{\xi,N}(t)u(t), \\ & \dot{\xi}(t) = \alpha\xi(t) + \beta p(t), \\ & z_N(0) = z_{N0},\ \xi(0) = \xi_0, \end{aligned} \tag{AP}$$

where $J_N(z_N, u)$ represents the finite-dimensional approximation of the cost of DPS regulation such that

$$\begin{aligned} J_N(z_N, u) = {} & \frac{1}{2}\int_{t_0}^{t_f} \Big(z_N(t)^T Q_N z_N(t) + u(t)^T R u(t)\Big)\mathrm{d}t \\ & + \frac{1}{2} z_N(t_f)^T Q_{fN} z_N(t_f). \end{aligned}$$

Remark 5: Problem (AP) is well-posed because the right-hand sides of the dynamics are Lipschitz continuous and, hence, there exists a unique solution of the differential equations [18].

Remark 6: Problem (AP) is not an LQR because $B_{\xi,N}(\cdot)$ is not linear in $\xi(\cdot)$, i.e., $B_{\xi_1,N}(t) + B_{\xi_2,N}(t) \neq B_{\xi_1+\xi_2,N}(t)$ for all $\xi_1(t), \xi_2(t) \in \mathbb{R}^m$ and for all t.

A. Gradient-descent method for the approximate integrated problem

To find a local minimum of (AP), we use Pontryagin's maximum principle, which establishes the following dynamics of the costates $\lambda(\cdot) \in \mathbb{R}^N$ and $\mu(\cdot) \in \mathbb{R}^m$ associated with $z_N(\cdot)$ and $\xi(\cdot)$, respectively:

$$\dot{\lambda}(t) = -A_N^T\lambda(t) - Q_N z_N(t), \tag{15a}$$

$$\dot{\mu}(t) = -\alpha^T\mu(t) - \kappa\xi(t) - \Big[\frac{\partial B_{\xi,N}(t)u(t)}{\partial\xi}\Big]^T\lambda(t), \tag{15b}$$

with terminal conditions:

$$\lambda(t_f) = Q_{fN} z_N(t_f), \tag{16a}$$

$$\mu(t_f) = \kappa_f(\xi(t_f) - \xi_f). \tag{16b}$$

The jointly optimal control $u^*(\cdot)$ and guidance $p^*(\cdot)$ satisfy

$$Ru^*(t) + B^T_{\xi^*,N}(t)\lambda^*(t) = 0, \tag{17a}$$

$$\gamma p^*(t) + \beta\mu^*(t) = 0. \tag{17b}$$

To obtain optimal control and guidance, we have to find a pair of states $(z_N^*(\cdot), \xi^*(\cdot))$ and costates $(\lambda^*(\cdot), \mu^*(\cdot))$ that satisfies (10), (2), and (15) with initial conditions (11), (3), and terminal condition (16). We adopt the gradient-descent method [19], [20] to numerically solve for $z_N^*(\cdot)$, $\xi^*(\cdot)$, $\lambda^*(\cdot)$, and $\mu^*(\cdot)$ that yield a local minimum of (AP).

B. Galerkin approximation

We use the Galerkin approximation scheme [21], which satisfies (9) [22], to approximate the infinite-dimensional system. We choose the standard first-order B-splines on the interval Ω, denoted by $\{\phi_i\}_{i=1}^N$, as the basis functions that span $\mathcal{H}_N$, where for $i = 1, 2, \ldots, N$,

$$\begin{aligned} \phi_i(x) = 1 - |(N+1)x - i|, & \\ \text{if } \frac{x - \Omega_l}{\Omega_r - \Omega_l} \in \Big[\frac{i-1}{N+1}, \frac{i+1}{N+1}\Big], & \end{aligned} \tag{18}$$

and $\phi_i(x) = 0$ otherwise, where Ω_l and Ω_r denote the left and right boundary of Ω, respectively. Let

$$\Phi_N(\cdot) = \big[\phi_1(\cdot), \phi_2(\cdot), \ldots, \phi_N(\cdot)\big]^T. \tag{19}$$

Consequently, the finite-dimensional approximation $z_N(t)$ of $\mathcal{Z}(t)$ satisfies

$$\mathcal{Z}(t) = \Phi_N^T z_N(t), \tag{20}$$

where the equality holds in the weak sense such that $\langle\mathcal{Z}(t), \psi\rangle = \langle\Phi_N^T z_N(t), \psi\rangle$ holds for any smooth test function $\psi \in \mathcal{H}$.

The Galerkin approximation takes the basis functions $\{\phi_i\}_{i=1}^N$ to be the test function. Let $M_N \in \mathbb{R}^{N\times N}$, $L_N \in \mathbb{R}^{N\times N}$ be such that

$$M_N = \int_\Omega \frac{\mathrm{d}\Phi_N(x)}{\mathrm{d}x}\frac{\mathrm{d}\Phi_N^T(x)}{\mathrm{d}x}\mathrm{d}x, \tag{21}$$

$$L_N = \int_\Omega \Phi_N(x)\Phi_N^T(x)\mathrm{d}x. \tag{22}$$

Now, the finite-dimensional approximation of (4) is

$$L_N\dot{z}_N(t) = -M_N z_N(t) + \bar{B}_{\xi,N}(t)u(t), \tag{23}$$

$$z_N(0) = \int_\Omega z_0(x)\Phi_N(x)\mathrm{d}x, \tag{24}$$

where $\bar{B}_{\xi,N}(\cdot)$ is an $N \times m$-dimensional matrix whose entry on the ith row and jth column is

$$[\bar{B}_{\xi,N}(t)]_{i,j} = \int_\Omega b_j\delta(x - \xi_j(t))\phi_i(x)\mathrm{d}x = b_j\phi_i(\xi_j(t)).$$

Hence, the parameters A_N and $B_{\xi,N}(\cdot)$ in (10) are such that

$$A_N = -(L_N)^{-1}M_N, \quad B_{\xi,N}(\cdot) = (L_N)^{-1}\bar{B}_{\xi,N}(\cdot). \tag{25}$$

Let $\mathcal{Q}$ and $\mathcal{Q}_f$ have kernel representations where there exist square-integrable functions $q : \Omega\times\Omega \to \mathbb{R}$ and $q_f : \Omega\times\Omega \to \mathbb{R}$, such that for any $\psi \in \mathcal{H}$ and $x \in \Omega$

$$\mathcal{Q}\psi(x) = \int_\Omega q(x, y)\psi(y)\mathrm{d}y, \tag{26}$$

$$\mathcal{Q}_f\psi(x) = \int_\Omega q_f(x, y)\psi(y)\mathrm{d}y. \tag{27}$$

To compute the optimal feedback control $\bar{u}^*$, assume that operator $\mathcal{S}$ admits a kernel representation $s : \Omega\times\Omega\times$

$[0,t_f] \to \mathbb{R}$ such that for every $x \in \Omega$, $t \in [0,t_f]$, and $\psi \in \mathcal{H}$,

$$\mathcal{S}(t)\psi(x) = \int_\Omega s(x,y,t)\psi(y)\mathrm{d}y. \tag{28}$$

The operator $\mathcal{S}$ admits a kernel representation if $\mathcal{S}$ is a Hilbert-Schmidt operator on $L^2(\Omega)$ [23].

Now, the differential Riccati equation with respect to s is

$$\begin{aligned} s_t(x,y,t) =& - s_{xx}(x,y,t) - s_{yy}(x,y,t) - q(x,y) \\ &+ s(x,\xi^*(t),t)R^{-1}s(\xi^*(t),y,t), \end{aligned} \tag{29}$$

$$s(x,y,t_f) = q_f(x,y). \tag{30}$$

Similar to (20), the Galerkin approximation of kernel s is spanned by bivariate B-spline basis $\{\phi_i(x)\phi_j(y)\}_{i,j=1}^N$ with the coordinates denoted by an $N \times N$-dimensional real matrix $S_N(\cdot)$ such that $s(x,y,t) = \Phi_N^T(x)S_N(t)\Phi_N(y)$. This equality holds in the weak sense such that, for any ψ in a Hilbert space with spatial domain $\Omega \times \Omega$,

$$\int_\Omega\int_\Omega [s(x,y,t) - \Phi_N^T(x)S_N(t)\Phi_N(y)]\psi(x,y)\mathrm{d}x\mathrm{d}y = 0.$$

The finite-dimensional approximation of (29) is

$$\begin{aligned} \dot{S}_N(t) =& - A_N S_N(t) - S_N(t)A_N^T - (L_N)^{-1}Q_N(L_N)^{-1} \\ &+ S_N(t)\bar{B}_{\xi^*,N}(t)R^{-1}\bar{B}_{\xi^*,N}^T(t)S_N(t), \end{aligned} \tag{31}$$

$$S_N(t_f) = Q_{fN}, \tag{32}$$

where $q(x,y) = \Phi_N^T(x)Q_N\Phi_N(y)$ and $q_f(x,y) = \Phi_N^T(x)Q_{fN}\Phi_N(y)$ hold in the weak sense.

The optimal state feedback control (subject to the given trajectory ξ of the mobile actuators) is

$$\bar{u}^* = -R^{-1}\bar{B}_{\xi^*,N}^T(t)S_N(t)L_N z_N(t). \tag{33}$$

C. Numerical example

We use the following values in a numerical example:

$$\begin{gathered} \Omega = [0,1], N = 50, m = 4, t_f = 1, \\ \alpha = 0, \beta = I_4, U = \mathbb{R}^m, R = 0.1I_4, \\ \gamma = 0.1I_4, \kappa = 0, \kappa_f = I_4, z_0(x) = 4x - 4x^2, \\ q(x,y) = 5\mathbb{1}(x=y), q_f(x,y) = \mathbb{1}(x=y), a = 0.1, \\ b_i = 1, [\xi(0)]_i = 0.01, [\xi_f]_i = 0.2i, \text{ for } i = 1,2,\dots,m, \end{gathered}$$

where $\mathbb{1}(x=y) = 1$ if $x = y$, and $\mathbb{1}(x=y) = 0$ if $x \neq y$. The forward propagation of z_N and ξ and backward propagation of λ and μ are computed using the Runge-Kutta method. The same method is also applied to propagate S_N in (31).

To demonstrate the performance of the optimal feedback control $\bar{u}^*$ subject to the optimal trajectory ξ^*, we define semi-naive control u_{sn} and naive control u_{n} as follows: both the semi-naive control and naive control dispense the actuation input whose magnitude is the unit negative value of the PDE at the locations of the actuators, i.e.,

$$u_{\text{sn}}(t) = -z_{\text{sn}}(\xi^*(t),t), \qquad u_{\text{n}}(t) = -z_{\text{n}}(\xi_{\text{n}}(t),t). \tag{34}$$

The semi-naive actuators follow the optimal trajectory ξ^*, whereas the naive actuators follow the trajectory ξ_{n}, which moves at a constant speed from ξ_0 to ξ_f.

In the simulation, a mobile pointwise disturbance $0.3\delta_{\xi_d}(t)$, whose trajectory is $\xi_d(t) = 0.5 - 0.49\sin(4\pi t)$, is added to the right-hand side of the dynamics (1). The same type of disturbance has been applied in [10]. Table I shows the cost breakdown of all the control and guidance in comparison. The optimal feedback control yields a smaller cost than the optimal open-loop control due to the capability of feedback control in rejecting disturbances. Simulations with a disturbance-free model (not shown) yield identical total cost for optimal open-loop control and optimal feedback control, which justifies the correctness of the synthesis. Fig. 1 compares the norm of the disturbed DPS state regulated by the control listed in Table I and zero control input $u \equiv 0$. As can be seen, the disturbed DPS is effectively regulated using optimal feedback control. Fig. 2 shows the optimal actuation input $\bar{u}^*$ and optimal trajectory ξ^* of each actuator. Actuator 4 dispenses the most actuation input to the DPS, compared with that by other actuators, because it reacts to the peak of the DPS before other actuators reach it. Fig. 3 shows the spatiotemporal distribution of the disturbed DPS with optimal feedback control $\bar{u}^*$ dispensed by actuators following the optimal trajectory ξ^*.

TABLE I: Cost breakdown of control and guidance in comparison. All costs are normalized with respect to the total cost of the naive control and guidance.

Control (C) and Guidance (G)			Cost		
	C	G	J_N	J_{m}	Total
opt. feedback	$\bar{u}^*$	ξ^*	41.3%	19.6%	60.9%
opt. open-loop	u^*	ξ^*	46.0%	19.6%	65.7%
semi-naive	u_{sn}	ξ^*	61.4%	19.6%	81.0%
naive	u_{n}	ξ_{n}	89.6%	10.4%	100.0%

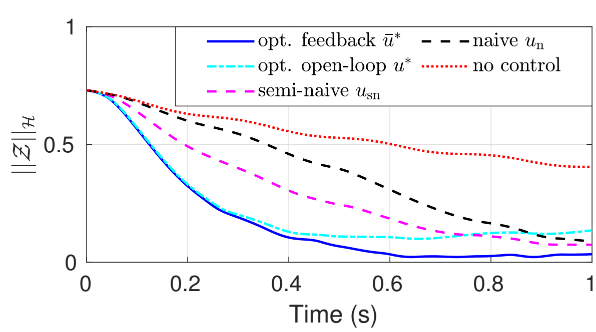

Fig. 1: Norm of the disturbed DPS state using different control and guidance

V. Conclusion

This paper proposes a guidance and control scheme that steers a team of mobile actuators to regulate a DPS modeled by a 1D diffusion process using an optimal control method. Specifically, jointly optimal guidance of the mobile actuators and control of the DPS are solved such that an integrated cost function is minimized subject to the dynamics of the diffusion process and the dynamics of the mobile actuators. We show that optimal guidance of the integrated problem,

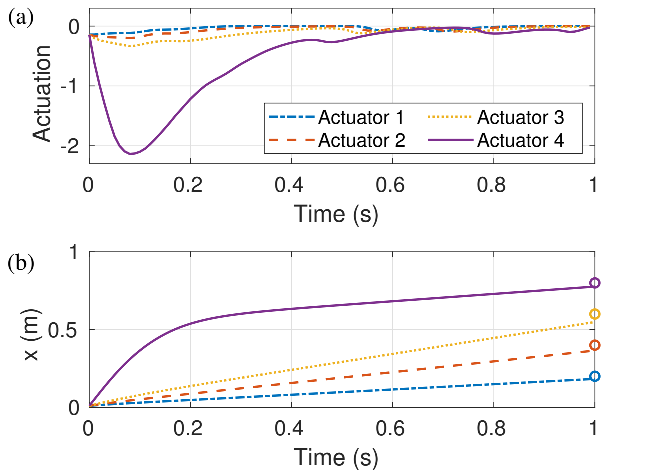

Fig. 2: Optimal actuation input (a) and trajectory (b) of each actuator. The circles in (b) indicate the desired terminal locations.

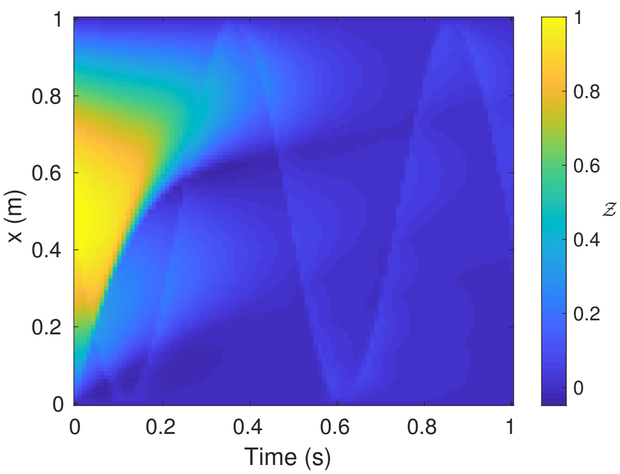

Fig. 3: Spatiotemporal distribution of the perturbed DPS. The sinusoidal curve shows the pointwise disturbance following the trajectory ξ_d.

under certain conditions, restricts the mobile actuators to the spatial domain even without explicit constraints. A gradient-descent method is applied to solve a finite-dimensional approximation of the integrated problem. We also provide a synthesis of the optimal feedback control with the jointly optimal trajectory of the mobile actuators using LQR theory. Lastly, we demonstrate our guidance and control in a numerical example where the Galerkin approximation scheme is applied for approximating the infinite-dimensional DPS.

Ongoing and future work includes establishing conditions under which the guidance solved from the approximated problem (AP) converges to the optimal guidance of (P); developing more efficient numerical methods to solve the integrated problem; extending the current work to a diffusion process in a 2D spatial domain; establishing a framework of simultaneous estimation and regulation of a DPS with a team of mobile sensor-actuators.

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