Three-Dimensional Motion Coordination in a Spatiotemporal Flowfield

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***Abstract*—Decentralized algorithms to stabilize three-dimensional formations of unmanned vehicles in a flowfield that varies in space and time have applications in environmental monitoring in the atmosphere and ocean. In this note, we provide a Lyapunov-based control design to steer a system of self-propelled particles traveling in three dimensions at a constant speed relative to a spatiotemporal flowfield. We assume that the flow is known locally to each particle and that it does not exceed the particle speed. Multiple particles can be steered to form three-dimensional parallel or helical formations in a flowfield. Also presented are motion coordination results for a special case of the three-dimensional model in which the particles travel in a circular formation on the surface of a rotating sphere.**

I. Introduction

Decentralized algorithms to stabilize collective motion in a three-dimensional flowfield that varies in space and time can be applied in many real-world scenarios [1], [2]. Previous work on collective motion in a flowfield has focused on a planar model of self-propelled particles [2]–[4], which is sufficient for studying motion coordination in a two-dimensional operational domain. Most prior work on non-planar collective motion has focused on flow-free models [5]–[7]. We provide a Lyapunov-based control design to steer a system of self-propelled particles traveling in three dimensions at a constant speed relative to a spatiotemporal flowfield. We assume that the flow is known locally to each particle and that it does not exceed the particle speed. Our model of three-dimensional motion coordination is motivated by unmanned vehicles that operate in a three-dimensional domain—such as underwater gliders [8] and long-endurance aircraft [9]. Motivated by constant-altitude/-depth surveys over spatial scales for which the curvature and/or rotation of Earth are relevant, we also study a special case of the three-dimensional model in which particles are constrained to the surface of a rotating sphere.

Our analysis extends [6] and [10], which describe decentralized strategies to steer a three-dimensional system of self-propelled particles in a flow-free environment. It also extends the three-dimensional analysis in [11], which includes a spatially variable, time-invariant flowfield. To model a spatiotemporal flowfield in three dimensions, we adapt the development of a planar framework for collective motion in a time-varying flow [2]. The spherical analysis extends [7], which introduced a flow-free spherical model. Additional prior work appears in [12] and an extended presentation of these results is available [13]. The study of motion coordination in an unknown flowfield or with turn-rate limits is ongoing [14] and outside the scope of this note, as is the study of flowfields that exceed the particle speed.

The contribution of this note is the Lyapunov-based design of decentralized control laws to stabilize moving formations of a three-dimensional, connected network of particles in a spatiotemporal flowfield. (This framework can also be extended to directed, time-varying communication topologies [15].) Previous results for three-dimensional motion coordination apply to flow-free models [6], [7]. We provide algorithms to stabilize parallel and helical formations in a three-dimensional flowfield that varies in space and time; we also provide an algorithm to stabilize circular formations in a spatiotemporal flowfield on the surface of a rotating sphere. (A parallel formation is a steady motion in which all of the particles travel in straight, parallel lines with arbitrary separation [6]. A helical formation is a steady motion in which the particles converge to circular helices with the same axis of rotation, radius of rotation, and pitch—the ratio of translational to rotational motion [6]; the along-axis separation is arbitrary. In a circular formation on a sphere all of the particles travel around a fixed circle [7].)

The note is organized as follows. In Section II, we describe a three-dimensional system of self-propelled particles in a spatiotemporal flowfield. In Section III, we describe a special case of the three-dimensional model in which the particles are constrained to travel on the surface of a rotating sphere. In Section IV, we provide control laws for the three-dimensional model to stabilize parallel formations in an arbitrary or prescribed direction. We also provide control laws for the three-dimensional model to stabilize helical formations with arbitrary center and pitch or prescribed center and pitch. In Section V, we provide control laws for the spherical model to stabilize circular formations with an arbitrary or prescribed center. Section VI summarizes the note and our ongoing work.

II. Particle Motion in Three Dimensions

The model studied in this section introduces a three-dimensional spatiotemporal flowfield to the flow-free particle model described in [5] and further studied in [10]. The flow-free model consists of N identical particles moving at unit speed in three dimensions. The position of particle $k \in \{1, \ldots, N\}$ is represented by $\mathbf{r}_k \in \mathbb{R}^3$ and its velocity relative to an inertial frame $\mathcal{I}$ by $\dot{\mathbf{r}}_k$. Control $\mathbf{u}_k = [w_k \ -h_k \ q_k]^T \in \mathbb{R}^3$ steers each particle by rotating its velocity about the unit vectors of a path frame, $\mathcal{C}_k = (k, \mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k)$, where

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$\mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k \in \mathbb{R}^3$. ($\mathcal{C}_k$ is a right-handed reference frame fixed to particle k such that the unit vector $\mathbf{x}_k$ points in the direction of the velocity of particle k.) The equations of motion are [5]

$$\begin{aligned} \dot{\mathbf{r}}_k &= \mathbf{x}_k,\ \dot{\mathbf{x}}_k = q_k\mathbf{y}_k + h_k\mathbf{z}_k \\ \dot{\mathbf{y}}_k &= -q_k\mathbf{x}_k + w_k\mathbf{z}_k,\ \dot{\mathbf{z}}_k = -h_k\mathbf{x}_k - w_k\mathbf{y}_k, \end{aligned} \tag{1}$$

where q_k (resp. h_k) represents the curvature control of the kth particle about the $\mathbf{y}_k$ (resp. $\mathbf{z}_k$) axis. The control w_k rotates the unit vectors $\mathbf{y}_k$ and $\mathbf{z}_k$ about the $\mathbf{x}_k$ axis.

The dynamics in (1) represent a control system on the Lie group $SE(3)$ [5], [10] and, consequently, can be expressed as $\dot{g}_k = g_k\hat{\boldsymbol{\xi}}_k = \begin{bmatrix} R_k & \mathbf{r}_k \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} \hat{\mathbf{u}}_k & \mathbf{e}_1 \\ \mathbf{0} & 0 \end{bmatrix}$, where $R_k = [\mathbf{x}_k\ \mathbf{y}_k\ \mathbf{z}_k]$ and $\mathbf{e}_1 = [1\ 0\ 0]^T$. The matrix $\hat{\boldsymbol{\xi}}_k \in \mathfrak{se}(3)$ is an element of the Lie algebra of $SE(3)$ and $\hat{\mathbf{u}}_k$ is the 3×3 skew-symmetric matrix that represents an element of $\mathfrak{so}(3)$, the Lie algebra of $SO(3)$. The dynamics are invariant under rigid motions in $SE(3)$ provided the controls $\mathbf{u}_k$ depend only on shape variables—relative positions and relative orientations of the path frames $\mathcal{C}_k$, $k \in \{1,...,N\}$. In this case, a special set of solutions correspond to steady-state formations, called relative equilibria [5], [10], in which the position and path-frame orientation of each particle is fixed with respect to the position and path-frame orientation of every other particle. The relative equilibria of model (1) are parallel formations, helical formations, and circular formations (helical formations with zero pitch) [5]. Algorithms to stabilize these formations in the flow-free model (1) are provided in [6]. Section IV provides algorithms to stabilize these types of formations in a three-dimensional, spatiotemporal flowfield.

Next we introduce a model of N particles traveling in a three-dimensional flowfield that varies in space and time. The instantaneous velocity of the flow at $\mathbf{r}_k$ is denoted by $\mathbf{f}_k(t) = \mathbf{f}(\mathbf{r}_k, t)$. Expressed in vector components with respect to the path frame C_k, we have $\mathbf{f}_k(t) = p_k(t)\mathbf{x}_k + t_k(t)\mathbf{y}_k + v_k(t)\mathbf{z}_k$. We make the following three assumptions about the flowfield: (A1) the components of $\mathbf{f}_k(t)$ expressed in $\mathcal{C}_k$ are known by particle k at time t; (A2) the flow speed is less than the particle speed relative to the flow, i.e., $||\mathbf{f}_k(t)|| < 1\ \forall\ k, t$; and (A3) $\mathbf{f}_k(t)$ is differentiable in $\mathbf{r}_k$ and t. The second assumption ensures that a particle can always make forward progress as measured from an inertial frame. The inertial velocity of particle k is the sum of its velocity relative to the flow and the velocity of the flow, i.e., $\dot{\mathbf{r}}_k = \mathbf{x}_k + \mathbf{f}_k(t) = (1 + p_k(t))\,\mathbf{x}_k + t_k(t)\mathbf{y}_k + v_k(t)\mathbf{z}_k$.

We associate frame $\mathcal{C}_k$ with motion relative to the flow-field. We can express the dynamics (1) relative to an inertial (or ground-fixed) frame using a second path frame, $\mathcal{D}_k = (k, \tilde{\mathbf{x}}_k, \tilde{\mathbf{y}}_k, \tilde{\mathbf{z}}_k)$. $\mathcal{D}_k$ is aligned with the inertial velocity of particle k, i.e., $\tilde{\mathbf{x}}_k$ is parallel to $\dot{\mathbf{r}}_k$. Let $s_k(t) = ||\mathbf{x}_k + \mathbf{f}_k(t)|| > 0$ denote the (time-varying) inertial speed of particle k. The dynamics expressed as components in frame $\mathcal{D}_k$ are

$$\begin{aligned} \dot{\mathbf{r}}_k &= s_k(t)\tilde{\mathbf{x}}_k,\ \dot{\tilde{\mathbf{x}}}_k = \tilde{q}_k\tilde{\mathbf{y}}_k + \tilde{h}_k\tilde{\mathbf{z}}_k \\ \dot{\tilde{\mathbf{y}}}_k &= -\tilde{q}_k\tilde{\mathbf{x}}_k + \tilde{w}_k\tilde{\mathbf{z}}_k,\ \dot{\tilde{\mathbf{z}}}_k = -\tilde{h}_k\tilde{\mathbf{x}}_k - \tilde{w}_k\tilde{\mathbf{y}}_k, \end{aligned} \tag{2}$$

where $\tilde{\mathbf{u}}_k = [\tilde{w}_k\ \ -\tilde{h}_k\ \ \tilde{q}_k]^T$ are the steering controls relative to frame $\mathcal{D}_k$. Note that the dynamics in (2) represent a control system on $SE(3)$, with R_k replaced by $\tilde{R}_k = [\tilde{\mathbf{x}}_k\quad \tilde{\mathbf{y}}_k\quad \tilde{\mathbf{z}}_k]$ and $\mathbf{e}_1$ replaced by $s_k(t)\mathbf{e}_1$. We will make use of the fact that (2) implies

$$\dot{\tilde{\mathbf{x}}}_k = \tilde{R}_k\tilde{\mathbf{u}}_k \times \tilde{\mathbf{x}}_k. \tag{3}$$

The definition of parallel and helical formations in three dimensions uses the concept of a twist [10], which is related to screw motion [16]. The operator $\vee$ generates a 6-dimensional vector that parametrizes a twist of the matrix $\hat{\tilde{\boldsymbol{\xi}}}_k$, where $\tilde{\boldsymbol{\xi}} = \hat{\tilde{\boldsymbol{\xi}}}_k^\vee = [s_k(t)\mathbf{e}_1\ \tilde{\mathbf{u}}_k]^T \in \mathbb{R}^6$ [16]. A constant screw motion is defined by the constant twist $\tilde{\boldsymbol{\xi}}_0 = [\tilde{\mathbf{v}}_0^T\ \ \boldsymbol{\omega}_0^T]^T \in \mathbb{R}^6$ [16]. When $\boldsymbol{\omega}_0 \neq 0$, the motion corresponds to rotation about an axis parallel to $\boldsymbol{\omega}_0$ and translation along $\boldsymbol{\omega}_0$. When $\boldsymbol{\omega}_0 = 0$, the motion corresponds to pure translation along $\tilde{\mathbf{v}}_0$. The pitch α_0 is the ratio of translational to rotational motion and its value in constant motion is $\alpha_0 = \boldsymbol{\omega}_0^T\tilde{\mathbf{v}}_0/\|\boldsymbol{\omega}_0\|^2$ if $\boldsymbol{\omega}_0 \neq 0$ and $\alpha_0 = \infty$ if $\boldsymbol{\omega}_0 = 0$ [16]. Following [10], we define a helical formation using the consensus variable (the superscript a indicates an inertial reference frame)

$$\tilde{\mathbf{v}}_k^a = \tilde{\mathbf{x}}_k + \mathbf{r}_k \times \boldsymbol{\omega}_0, \quad \boldsymbol{\omega}_0 \neq 0. \tag{4}$$

Using (3), the velocity of $\tilde{\mathbf{v}}_k^a$ along solutions of (2) is $\dot{\tilde{\mathbf{v}}}_k^a = (\tilde{R}_k\tilde{\mathbf{u}}_k - s_k(t)\boldsymbol{\omega}_0) \times \tilde{\mathbf{x}}_k$. The following two lemmas extend [10, Proposition 1] to helical and parallel motion in a spatiotemporal flowfield.

Lemma 1: The control $\tilde{\mathbf{u}}_k = \tilde{R}_k^T s_k(t)\boldsymbol{\omega}_0$ steers particle k in model (2) with time-varying flow $\mathbf{f}_k(t)$ along a circular helix with axis parallel to $\boldsymbol{\omega}_0$, radius $\|\boldsymbol{\omega}_0\|^{-1}$, and pitch $\alpha_k = \boldsymbol{\omega}_0^T\tilde{\mathbf{x}}_k/\|\boldsymbol{\omega}_0\|^2 \leq 1$. The quantity $\tilde{\mathbf{v}}_k^a$ defined in (4) is fixed. A helical formation of N particles is characterized by $\tilde{\mathbf{v}}_k^a = \tilde{\mathbf{v}}_j^a$ for all pairs $j, k \in \{1,...,N\}$.

Lemma 2: The control $\tilde{\mathbf{u}}_k = \mathbf{0}$ steers particle k in model (2) with time-varying flow $\mathbf{f}_k(t)$ along a straight line such that $\tilde{\mathbf{x}}_k$ is fixed. A parallel formation of N particles is characterized by $\tilde{\mathbf{x}}_k = \tilde{\mathbf{x}}_j$ for all pairs $j, k \in \{1,...,N\}$.

Next we summarize the transformation between the flow-relative frame $\mathcal{C}_k$ and frame $\mathcal{D}_k$. This relationship is important because we design the steering controls $\tilde{\mathbf{u}}_k$ using (2) and the platform dynamics are presumed to obey (1); one needs to compute $\mathbf{u}_k$ from $\tilde{\mathbf{u}}_k$ to implement the algorithm. Using $\dot{\mathbf{r}}_k = \mathbf{x}_k + \mathbf{f}_k(t)$ and (2), we have $\tilde{\mathbf{x}}_k = \frac{1+p_k(t)}{s_k(t)}\mathbf{x}_k + \frac{t_k(t)}{s_k(t)}\mathbf{y}_k + \frac{v_k(t)}{s_k(t)}\mathbf{z}_k$. Note $\tilde{\mathbf{x}}_k$ lies in the plane spanned by $\mathbf{x}_k$ and $\mathbf{f}_k(t)$. Let θ be the angle between $\mathbf{x}_k$ and $\tilde{\mathbf{x}}_k$ such that $0 \leq \theta \leq \pi$, which implies $\cos\theta = \mathbf{x}_k \cdot \tilde{\mathbf{x}}_k = s_k^{-1}(1 + p_k(t))$ and $\sin\theta = ||\mathbf{x}_k \times \tilde{\mathbf{x}}_k|| = s_k^{-1}\sqrt{t_k^2(t) + v_k^2(t)}$. Using these relations, it is possible to derive the transformation between frames $\mathcal{C}_k$ and $\mathcal{D}_k$. (Omitted due to length; see [13].)

Unlike in the flow-free model (1), the inertial speed $s_k(t)$ of particle k in model (2) is not constant—it depends on the flow and the direction of motion. The inertial speed of particle k is $s_k(t) = ||s_k(t)\tilde{\mathbf{x}}_k|| = \sqrt{(1 + p_k(t))^2 + t_k^2(t) + v_k^2(t)} > 0$, where $p_k(t)$, $t_k(t)$, and $v_k(t)$ are components of $\mathbf{f}_k(t)$ in $\mathcal{C}_k$. In order to integrate (2), we need the following expression for $s_k(t)$ in terms of the components of $\mathbf{f}_k(t)$ in $\mathcal{D}_k$.

Theorem 1: The inertial speed of particle k in model (2) with flow $\mathbf{f}_k(t)$ is

$$s_k(t) = \sqrt{1 - \|\tilde{\mathbf{x}}_k \times \mathbf{f}_k(t)\|^2} + \tilde{\mathbf{x}}_k \cdot \mathbf{f}_k(t). \tag{5}$$

The proof of Theorem 1 can be found in [13]. Note, (5) is used to integrate (2) and only requires knowledge of $\mathbf{f}_k(t)$ expressed in $\mathcal{D}_k$. However, to compute $\mathbf{u}_k$ from $\tilde{\mathbf{u}}_k$, we need to know $\mathbf{f}_k(t)$ in $\mathcal{C}_k$.

III. Particle Motion on a Rotating Sphere

We now constrain particle motion in the three-dimensional model to the surface of a rotating sphere. The model studied here extends the spherical model introduced in [7], which consists of N particles moving at a constant speed on the surface of a non-rotating sphere. We expand this framework by first introducing rotation to the sphere and then attaching a spatiotemporal flowfield to its surface. The flow-free model consists of N particles moving at unit speed on the surface of a sphere with radius $\rho_0 > 0$ and center O. The position of particle $k \in \{1, \ldots, N\}$ relative to O is represented by $\mathbf{r}_k$. A path frame $\mathcal{C}_k = (k, \mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k)$ is fixed to particle k such that the unit vector $\mathbf{x}_k$ points in the direction of the velocity of particle k, $\mathbf{z}_k = \rho_0^{-1}\mathbf{r}_k$ is orthogonal to the sphere at $\mathbf{r}_k$, and $\mathbf{y}_k$ completes the right-handed reference frame. A gyroscopic force steers each particle k on the surface of the sphere, modeled as a state-feedback control u_k that rotates the velocity of each particle about $\mathbf{z}_k$. The dynamics are [7]

$$\begin{array}{rcl}\dot{\mathbf{r}}_k & = & \mathbf{x}_k,\ \dot{\mathbf{x}}_k = u_k\mathbf{y}_k - \rho_0^{-1}\mathbf{z}_k \\ \dot{\mathbf{y}}_k & = & -u_k\mathbf{x}_k,\ \dot{\mathbf{z}}_k = \rho_0^{-1}\mathbf{x}_k.\end{array} \tag{6}$$

One obtains (6) directly from (1) by substituting $\mathbf{u}_k = [0\ 0\ u_k]^T$ into (1) and incorporating the radius of the sphere. As such, the dynamics in (6) represent a control system on the Lie group $SE(3)$ [5], [17]. Alternatively, since $\mathbf{r}_k$ is parallel to $\mathbf{z}_k$, the dynamics evolve on $SO(3)$ according to $\dot{R}_k = R_k\hat{\boldsymbol{\eta}}_k = \begin{bmatrix}\mathbf{x}_k & \mathbf{y}_k & \mathbf{z}_k\end{bmatrix}\hat{\boldsymbol{\eta}}_k$, where $\hat{\boldsymbol{\eta}}_k \in \mathfrak{so}(3)$ is the 3×3 skew-symmeric matrix generated from $\boldsymbol{\eta}_k = [0\ \rho_0^{-1}\ u_k]^T$.

We first extend the framework described in [7] by adding rotation to the sphere, which introduces the Coriolis acceleration $\mathbf{a}_k^{cor} = -2\boldsymbol{\omega}_1 \times \dot{\mathbf{r}}_k$, where $\boldsymbol{\omega}_1$ is the angular velocity of the sphere and $\dot{\mathbf{r}}_k$ is the velocity relative to the surface of the sphere. To derive the particle dynamics on a rotating sphere, we use a spherical coordinate system consisting of the azimuth angle θ_k, the polar angle ϕ_k, and the (fixed) radius ρ_0. $\mathcal{I} = (O, \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ is an inertial reference frame with origin O at the center of the sphere. We assume that $\boldsymbol{\omega}_1 = \omega_1\mathbf{e}_z$, so that the sphere rotates at constant angular rate ω_1 about $\mathbf{e}_z$.

We introduce four additional reference frames. Frame $\mathcal{I}' = (O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is fixed to the sphere and differs from $\mathcal{I}$ by a rotation of $\omega_1 t$ about $\mathbf{e}_3 = \mathbf{e}_z$, where t is time. Frame $\mathcal{A}_k = (O, \mathbf{a}_{1_k}, \mathbf{a}_{2_k}, \mathbf{a}_{3_k})$, $k \in \{1, \ldots, N\}$, differs from $\mathcal{I}'$ by a rotation of θ_k about $\mathbf{a}_{3_k} = \mathbf{e}_3$. Frame $\mathcal{B}_k = (O, \mathbf{e}_{\phi_k}, \mathbf{e}_{\theta_k}, \mathbf{e}_{r_k})$ differs from $\mathcal{A}_k$ by a rotation of ϕ_k about $\mathbf{e}_{\theta_k} = \mathbf{a}_{2_k}$. The unit vector $\mathbf{e}_{r_k}$ points from O to the position $\mathbf{r}_k$ of particle k. The fourth frame, $\mathcal{C}_k = (k, \mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k)$, differs from $\mathcal{B}_k$ by a rotation of the orientation angle γ_k about $\mathbf{z}_k = \mathbf{e}_{r_k}$. The origin of $\mathcal{C}_k$ is attached to particle k and the unit vector $\mathbf{x}_k$ points in the direction of motion of particle k relative to the sphere-fixed frame $\mathcal{I}'$. The angular velocity ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{C}_k}$ of $\mathcal{C}_k$ with respect to the inertial frame $\mathcal{I}$ is ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{C}_k} = (\omega_1 + \dot{\theta}_k)\mathbf{a}_{3_k} + \dot{\phi}_k\mathbf{e}_{\theta_k} + \dot{\gamma}\mathbf{z}_k$, where $\mathbf{a}_{3_k} = -\sin\phi_k\cos\gamma_k\mathbf{x}_k + \sin\phi_k\sin\gamma_k\mathbf{y}_k + \cos\phi_k\mathbf{z}_k$ and $\mathbf{e}_{\theta_k} = \sin\gamma_k\mathbf{x}_k + \cos\gamma_k\mathbf{y}_k$ [7].

The inertial acceleration due to the Coriolis effect is $\mathbf{a}_k^{cor} = -2{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{C}_k} \times \dot{\mathbf{r}}_k = -2\omega_1(\cos\phi_k\mathbf{y}_k - \sin\gamma_k\mathbf{z}_k)$. The Coriolis acceleration contributes a fictional force, $\mathbf{F}_k^{cor} = \mathbf{a}_k^{cor}$ (assuming the particles have unit mass). The force $\mathbf{F}_k = -N_k\mathbf{z}_k + u_k\mathbf{y}_k$ on particle k in the non-rotating sphere is the sum of the normal force $N_k = \rho_0^{-1}$ that acts orthogonally to the surface of the sphere and the steering force u_k that acts tangentially to the surface of the sphere and orthogonally to $\mathbf{x}_k$. The total apparent force in the rotating sphere is $\mathbf{F}_k^{tot} = \mathbf{F}_k + \mathbf{F}_k^{cor} = (2\omega_1\sin\gamma_k - N_k)\mathbf{z}_k + (u_k - 2\omega_1\cos\phi_k)\mathbf{y}_k$. Comparing the $\mathbf{y}_k$ components of $\mathbf{F}_k^{tot}$ and $\mathbf{F}_k$, we observe that the control u_k is augmented by $-2\omega_1\cos\phi_k$. Since $\cos\phi_k = \mathbf{z}_k\cdot\mathbf{e}_3 = z_{k_3}$, we define the effective control [12]

$$\nu_k = u_k - 2\omega_1 z_{k_3}. \tag{7}$$

Under the effective control (7), the dynamics of particle k relative to the sphere-fixed frame $\mathcal{I}'$ are

$$\begin{array}{rcl}\dot{\mathbf{r}}_k & = & \mathbf{x}_k,\ \dot{\mathbf{x}}_k = \nu_k\mathbf{y}_k - \rho_0^{-1}\mathbf{z}_k \\ \dot{\mathbf{y}}_k & = & -\nu_k\mathbf{x}_k,\ \dot{\mathbf{z}}_k = \rho_0^{-1}\mathbf{x}_k.\end{array} \tag{8}$$

Thus, if we use (8) to design ν_k, we can cancel the Coriolis acceleration by using (7) to compute u_k.

The only relative equilibrium of the closed-loop dynamics on the (non-rotating) sphere is circular motion with a common radius and axis and direction of rotation [7]. Algorithms to stabilize circular formations in the flow-free model (6) are provided in [7]. We provide algorithms to stabilize circular formations in a spatiotemporal flowfield on a rotating sphere in Section V. A circular trajectory on one side of the surface of a sphere can be described as the intersection of the sphere and a right circular cone whose axis of rotation passes through the center of the sphere and whose apex is outside the sphere. The position $\mathbf{c}_k$ (relative to O) of the apex of the cone is $\mathbf{c}_k = \mathbf{r}_k + \omega_0^{-1}\mathbf{y}_k$, where $\omega_0 \neq 0$ and the chordal radius of the circle is $|\omega_0|^{-1}$ [7]. The velocity of $\mathbf{c}_k$ along solutions of (8) is $\dot{\mathbf{c}}_k \triangleq \frac{{}^{\mathcal{I}'}d}{dt}\mathbf{c}_k = (1 - \omega_0^{-1}\nu_k)\mathbf{x}_k$. If $\mathbf{c}_k = \mathbf{c}_j$ for all pairs j and k, we call this motion a circular formation with center $\mathbf{c}_k$. The following extends [7, Proposition 2] to a rotating sphere.

Lemma 3: The control $\nu_k = \omega_0$ steers particle k in model (8) around a circle such that the center $\mathbf{c}_k$ is fixed. A circular formation is characterized by $\mathbf{c}_k = \mathbf{c}_j$ for all pairs $j, k \in \{1, ..., N\}$.

We now study the case of N particles traveling on a rotating sphere in a spatiotemporal flowfield. The velocity of the flow at the position $\mathbf{r}_k$ is represented by $\mathbf{f}_k(t) = \mathbf{f}(\mathbf{r}_k, t)$, which can be decomposed into vector components in frame $\mathcal{C}_k$ as $\mathbf{f}_k(t) = p_k(t)\mathbf{x}_k + t_k(t)\mathbf{y}_k$. (There is no flow orthogonal to the surface of the sphere.) Assumptions (A1)–(A3) apply here. Adding $\mathbf{f}_k(t)$ to the time derivative of the position of the particle model in (8) we obtain $\dot{\mathbf{r}}_k = (1 + p_k(t))\mathbf{x}_k + t_k(t)\mathbf{y}_k$. Since $\dot{\mathbf{z}}_k$ is parallel to $\dot{\mathbf{r}}_k$, i.e., $\dot{\mathbf{z}}_k = \rho_0^{-1}\dot{\mathbf{r}}_k$, and the dynamics evolve on $SO(3)$, the remaining equations of motion can be found from $\dot{R}_k = R_k\hat{\tilde{\boldsymbol{\eta}}}_k$, where $R_k \triangleq [\mathbf{x}_k\ \ \mathbf{y}_k\ \ \mathbf{z}_k]$, $\tilde{\boldsymbol{\eta}}_k = [-\rho_0^{-1}t_k(t)\ \rho_0^{-1}(1 + p_k(t))\ \nu_k]$, and ν_k is defined in (7).

In order to find a control law to stabilize a formation in a spatiotemporal flow, we express the dynamics in frame $\mathcal{D}_k = (k, \tilde{\mathbf{x}}_k, \tilde{\mathbf{y}}_k, \tilde{\mathbf{z}}_k)$. The motion of a particle relative to the sphere is the sum of the particle motion relative to the flow and the motion of the flow relative to the sphere. We choose $\tilde{\mathbf{x}}_k$ to be parallel to $\dot{\mathbf{r}}_k$. The speed of particle k relative to $\mathcal{I}'$ is denoted $s_k(t)$. The dynamics are

$$\begin{aligned}\dot{\mathbf{r}}_k &= s_k(t)\tilde{\mathbf{x}}_k,\ \dot{\tilde{\mathbf{x}}}_k = \tilde{\nu}_k\tilde{\mathbf{y}}_k - \rho_0^{-1}s_k(t)\tilde{\mathbf{z}}_k \\ \dot{\tilde{\mathbf{y}}}_k &= -\tilde{\nu}_k\tilde{\mathbf{x}}_k,\ \dot{\tilde{\mathbf{z}}}_k = \rho_0^{-1}s_k(t)\tilde{\mathbf{x}}_k,\end{aligned} \tag{9}$$

where $\tilde{\nu}_k$ is the control input in frame $\mathcal{D}_k$. In $\mathcal{D}_k$, we have

$$\tilde{\mathbf{c}}_k = \mathbf{r}_k + \omega_0^{-1}\tilde{\mathbf{y}}_k. \tag{10}$$

The velocity of $\tilde{\mathbf{c}}_k$ along solutions of (9) is $\dot{\tilde{\mathbf{c}}}_k = (s_k(t) - \omega_0^{-1}\tilde{\nu}_k)\tilde{\mathbf{x}}_k$. The following extends Lemma 3 to motion on a rotating sphere in a spatiotemporal flowfield.

Lemma 4: The control $\tilde{\nu}_k = \omega_0 s_k(t)$ steers particle k in model (9) with time-varying flow $\mathbf{f}_k(t)$ around a circle such that the center $\tilde{\mathbf{c}}_k$ defined in (10) is fixed. A circular formation is characterized by $\tilde{\mathbf{c}}_k = \tilde{\mathbf{c}}_j$ for all pairs $j,\ k \in \{1, ..., N\}$.

The relationship between frames $\mathcal{C}_k$ and $\mathcal{D}_k$ is important because we design $\tilde{\nu}_k$ using (9) and the platform dynamics are presumed to obey (8); one needs to compute ν_k from $\tilde{\nu}_k$. Since $\tilde{\mathbf{z}}_k = \mathbf{z}_k$, we use $\dot{R}_k = R_k\hat{\tilde{\boldsymbol{\eta}}}_k$ and (9) to find $\tilde{\mathbf{x}}_k = s_k(t)^{-1}\left[(1 + p_k(t))\,\mathbf{x}_k + t_k(t)\mathbf{y}_k\right]$. The relationship between $\tilde{\nu}_k$ and ν_k is found by taking the time derivative of $\tilde{\mathbf{x}}_k$ [13]. The inertial speed is $s_k(t) = \sqrt{(1 + p_k(t))^2 + t_k^2(t)} > 0$, where $p_k(t)$ and $t_k(t)$ are the components of $\mathbf{f}_k(t)$ in $\mathcal{C}_k$. To integrate (9) we express $s_k(t)$ in terms of the components of $\mathbf{f}_k(t)$ in frame $\mathcal{D}_k$ using Theorem 1.

IV. Motion Coordination in Three Dimensions

In this section, we derive decentralized control laws that stabilize three-dimensional parallel and helical formations in a spatiotemporal flowfield, following the flow-free approach provided in [5] and [10]. We use a graph G to represent the particle communication topology, which we assume to be connected, undirected, and time-invariant. (The extension to directed, time-varying topologies follows [15].)

A parallel formation is a steady motion in which all of the particles travel in straight, parallel lines with arbitrary separation. We first provide a control law to stabilize parallel formations in an arbitrary direction of motion and then a control law that prescribes the direction. Let $G = (\mathcal{N}, E)$, where $\mathcal{N} = \{1, ...N\}$, have edge set E and graph Laplacian L [18]. Using the matrix $\mathcal{L} \triangleq L \otimes I_3$, where $\otimes$ denotes the Kronecker product, we define the potential (we drop the subscript to denote a matrix collection of N elements)

$$S(\tilde{\mathbf{x}}) = \tfrac{1}{2}\tilde{\mathbf{x}}^T\mathcal{L}\tilde{\mathbf{x}} = \tfrac{1}{2}\textstyle\sum_{(j,k)\in E}\|\tilde{\mathbf{x}}_j - \tilde{\mathbf{x}}_k\|^2, \tag{11}$$

where $\tilde{\mathbf{x}} \triangleq [\tilde{\mathbf{x}}_1^T \cdots \tilde{\mathbf{x}}_N^T]^T$ and the summation is over all of the edges in G. The potential S is minimized by the set of parallel formations in accordance with Lemma 2.

Using (3) we find the time derivative of S along solutions of (2) to be $\dot{S} = \sum_{j=1}^N \dot{\tilde{\mathbf{x}}}_j \cdot \mathcal{L}_j\tilde{\mathbf{x}} = \sum_{j=1}^N \left(\tilde{R}_j\tilde{\mathbf{u}}_j \times \tilde{\mathbf{x}}_j\right)\cdot\mathcal{L}_j\tilde{\mathbf{x}}$, where $\mathcal{L}_k$ denotes three consecutive rows of $\mathcal{L}$ starting with row $3k-2$, $k \in \mathcal{N}$. Choosing the control law

$$\tilde{\mathbf{u}}_k = -K_0\tilde{R}_k^T\left(\tilde{\mathbf{x}}_k \times \mathcal{L}_k\tilde{\mathbf{x}}\right),\ K_0 > 0, \tag{12}$$

ensures that S is nonincreasing, since $\dot{S} = -K_0\sum_{j=1}^N(\tilde{\mathbf{x}}_j \times \mathcal{L}_j\tilde{\mathbf{x}} \times \tilde{\mathbf{x}}_j)\cdot\mathcal{L}_j\tilde{\mathbf{x}} = -K_0\sum_{j=1}^N||\tilde{\mathbf{x}}_j \times \mathcal{L}_j\tilde{\mathbf{x}}||^2 \leq 0$. The following extends [10, Theorem 1] to motion in a flowfield that varies in space and time.

Theorem 2: All solutions of the closed-loop model (2) with time-varying flow $\mathbf{f}_k(t)$, the speed $s_k(t)$ given by (5), and the control $\tilde{\mathbf{u}}_k$ by (12), converge to the set $\{\dot{S} = 0\}$, where S is defined in (11). The set of parallel formations is uniformly asymptotically stable and the direction of motion is determined by the initial conditions.

Proof: The closed-loop dynamics $\dot{\tilde{\mathbf{x}}}_k = \tilde{R}_k\tilde{\mathbf{u}}_k \times \tilde{\mathbf{x}}_k$ with $\tilde{\mathbf{u}}_k$ given by (12) are autonomous and independent of the flowfield. Since each $\tilde{\mathbf{x}}_k$ has unit length, the dynamics evolve on a compact space isomorphic to N copies of the two-sphere $\mathbb{S}^2$. The potential S is radially unbounded and positive-definite in the reduced space of relative velocities, and its time derivative satisfies $\dot{S} \leq 0$. By the invariance principle, all solutions converge to the largest invariant set in $\{\dot{S} = 0\}$, for which $\tilde{\mathbf{x}}_k \times \mathcal{L}_k\tilde{\mathbf{x}} = 0$, $k \in \mathcal{N}$. When $\mathcal{L}_k\tilde{\mathbf{x}} = 0\ \forall\ k$, the potential S is minimized, forming the set of parallel formations $\tilde{\mathbf{x}}_j = \tilde{\mathbf{x}}_k\ \forall j, k \in \mathcal{N}$. Since this set corresponds to the global minimum of S, it is asymptotically stable. ∎

Theorem 2 provides a decentralized algorithm to stabilize a parallel formation in a three-dimensional flowfield using control (12), as illustrated for all-to-all communication in Fig. 1(a). The direction of motion of the formation is determined by the initial conditions. The flowfield is a three-dimensional adaptation of a Rankine vortex, with maximum flow speed occurring on a set of radii that increases with height. The vortex translates horizontally (see dashed arrows).

Next we provide a control algorithm that stabilizes parallel formations in a prescribed direction by introducing a virtual particle $k = 0$ that travels parallel to $\boldsymbol{\omega}_0 \neq 0$. The dynamics of the virtual particle are given by (2) with $\tilde{\mathbf{u}}_0 = 0$ and $\tilde{\mathbf{x}}_0(t) = \tilde{\mathbf{x}}_0(0) = \boldsymbol{\omega}_0/\|\boldsymbol{\omega}_0\|$. Let $G_0 = (\mathcal{N}_0, E_0)$, where $\mathcal{N}_0 = \{0, 1, ..., N\}$, denote a time-invariant and directed graph rooted to node 0. The edge set $E \subset E_0$ includes at least one link from particle 0 to a particle $k \in \mathcal{N}$. We define

$$S_0(\tilde{\mathbf{x}}) = \tfrac{1}{2}\tilde{\mathbf{x}}^T\mathcal{L}\tilde{\mathbf{x}} + \tfrac{1}{2}\textstyle\sum_{j=1}^N a_{j0}\|\tilde{\mathbf{x}}_j - \tilde{\mathbf{x}}_0\|^2, \tag{13}$$

where $a_{j0} = 1$ if there is information flow from particle 0 to particle $j \in \mathcal{N}$ and $a_{j0} = 0$ otherwise. Using (3), the time derivative of S_0 is $\dot{S}_0 = -\sum_{j=1}^N[\tilde{R}_j\tilde{\mathbf{u}}_j \times \tilde{\mathbf{x}}_j]\cdot[\mathcal{L}_j\tilde{\mathbf{x}} - a_{j0}(\tilde{\mathbf{x}}_j - \boldsymbol{\omega}_0/\|\boldsymbol{\omega}_0\|)]$. Choosing

$$\tilde{\mathbf{u}}_k = K_0\tilde{R}_k^T\tilde{\mathbf{x}}_k \times (\mathcal{L}_k\tilde{\mathbf{x}} - a_{k0}(\tilde{\mathbf{x}}_k - \boldsymbol{\omega}_0/\|\boldsymbol{\omega}_0\|)),\ K_0 > 0, \tag{14}$$

ensures that S_0 is non-increasing.

Corollary 1: All solutions of the closed-loop model (2) with time-varying flow $\mathbf{f}_k(t)$, the speed $s_k(t)$ given by (5), and the control $\tilde{\mathbf{u}}_k$ by (14), converge to the set $\{\dot{S}_0 = 0\}$, where S_0 is defined in (13). The set of parallel formations with direction of motion determined by $\boldsymbol{\omega}_0$ is uniformly asymptotically stable.

A helical formation is a steady motion in which all of the particles converge to circular helices with the same axis of rotation, radius, and pitch. We now provide control laws to stabilize helical formations with arbitrary pitch and center and prescribed pitch and center. The potential

$$Q(\tilde{\mathbf{v}}^a) = \tfrac{1}{2}\left(\tilde{\mathbf{v}}^a\right)^T\mathcal{L}\tilde{\mathbf{v}}^a = \tfrac{1}{2}\textstyle\sum_{(j,k)\in E}||\tilde{\mathbf{v}}_j^a - \tilde{\mathbf{v}}_k^a||^2, \tag{15}$$

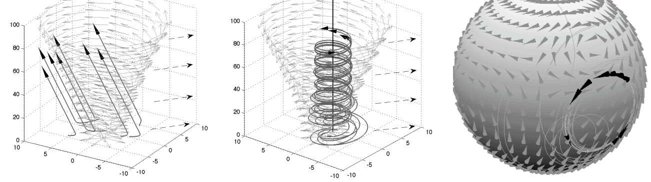

Fig. 1. Three-dimensional motion coordination in a spatiotemporal flowfield: parallel, helical, and circular formations.

where $\mathcal{L} \triangleq L \otimes I_3$ and $\tilde{\mathbf{v}}_k^a$ is defined in (4), is minimized by the set of helical formations in accordance with Lemma 1. Recall $\mathcal{L}_k$, $k \in 1, ..., N$, denotes three consecutive rows of $\mathcal{L}$ starting with row $3k-2$. Along solutions of (2), $Q = \sum_{j=1}^N \mathcal{L}_j \tilde{\mathbf{v}}^a \cdot (\dot{\tilde{\mathbf{x}}}_j + \dot{\mathbf{r}}_j \times \boldsymbol{\omega}_0)$, where $\dot{\tilde{\mathbf{x}}}_j$ given by (3). Choosing

$$\tilde{\mathbf{u}}_k = K_0 \tilde{R}_k^T \left[s_k(t)\boldsymbol{\omega}_0 + \mathcal{L}_k \tilde{\mathbf{v}}^a \times \tilde{\mathbf{x}}_k \right], \; K_0 > 0, \tag{16}$$

results in $\dot{Q} = -K_0 \sum_{j=1}^N \mathcal{L}_j \tilde{\mathbf{v}}^a \cdot (\tilde{\mathbf{x}}_j \times \mathcal{L}_j \tilde{\mathbf{v}}^a \times \tilde{\mathbf{x}}_j) = -K_0 \sum_{j=1}^N ||\tilde{\mathbf{x}}_j \times \mathcal{L}_j \tilde{\mathbf{v}}^a||^2 \leq 0$. The following result extends [10, Theorem 2] to motion in a spatiotemporal flowfield.

Theorem 3: All solutions of the closed-loop model (2) with time-varying flow $\mathbf{f}_k(t)$, the speed $s_k(t)$ given by (5), and the control $\tilde{\mathbf{u}}_k$ by (16), converge to the set $\{\dot{Q} = 0\}$, where Q is defined in (15). The set of helical formations with axis of rotation parallel to $\boldsymbol{\omega}_0$, radius $\|\boldsymbol{\omega}_0\|^{-1}$, and arbitrary pitch is uniformly asymptotically stable and the formation pitch and center are determined by the initial conditions.

Proof: The closed-loop dynamics (2) with $\tilde{\mathbf{u}}_k$ given in (16) depend on the time-varying speed $s_k(t)$. The proof follows from a pair of invariance-like theorems for nonautonomous systems [19, Theorems 8.4 and 8.5]. Q is radially unbounded and positive definite in the reduced space of the relative quantities $\tilde{\mathbf{v}}_k^a - \tilde{\mathbf{v}}_j^a \; \forall \, (j,k) \in E$. The time derivative of Q satisfies $\dot{Q} \leq 0$ and neither Q nor $\dot{Q}$ depend explicitly on time. By [19, Theorem 8.4] all solutions of the closed-loop model converge to the set $\{\dot{Q} = 0\}$, in which $\tilde{\mathbf{x}}_k \times \mathcal{L}_k \tilde{\mathbf{v}}^a = 0 \; \forall \, k$. The potential Q is minimized when $\mathcal{L}_k \tilde{\mathbf{v}}^a = 0 \; \forall \, k \in \mathcal{N}$, which implies the set of helical formations $\tilde{\mathbf{v}}_j = \tilde{\mathbf{v}}_k \; \forall \, k, j \in \mathcal{N}$ it is uniformly asymptotically stable. ■

Theorem 3 provides a method to stabilize helical formations in a three-dimensional flowfield. The pitch and center of a helical formation stabilized by control (16) are determined by the initial conditions. Note, the parameter $\boldsymbol{\omega}_0$ used in (16) prescribes a line parallel to the axis of rotation of the formation, but not necessarily the location of this axis (the formation center). To isolate helical formations with a prescribed pitch and center we introduce a virtual particle $k = 0$ with dynamics given by (2) with $\tilde{\mathbf{u}}_0 = s_k(t)\tilde{R}_k \boldsymbol{\omega}_0$, so that $\tilde{\mathbf{v}}_0^a$ is invariant for all time, i.e., $\tilde{\mathbf{v}}_0^a(t) = \tilde{\mathbf{v}}_0^a(0)$. The twist of the virtual particle, $\tilde{\boldsymbol{\xi}}_0^a = [(\tilde{\mathbf{v}}_0^a)^T, \; \boldsymbol{\omega}_0^T]^T$, is constant and its corresponding pitch is $\tilde{\mathbf{v}}_0^a \cdot \boldsymbol{\omega}_0 / \|\boldsymbol{\omega}_0\|^2$ [10], [16].

To isolate helical formations with a prescribed center and pitch we use the center $\tilde{\mathbf{v}}_0^a$ and the pitch $\alpha_0 \in [0, 1)$ (a pitch of $\alpha_0 = 0$ results in a three-dimensional circular formation). We define the potentials [10]

$$\begin{aligned} Q_{pitch}(\tilde{\mathbf{v}}^a) &= \tfrac{1}{2} \textstyle\sum_{j=1}^N a_{j0} (\tilde{\mathbf{v}}_j^a \cdot \boldsymbol{\omega}_0 / \|\boldsymbol{\omega}_0\| - \alpha_0)^2 \\ Q_{center}(\tilde{\mathbf{v}}^a) &= \tfrac{1}{2} \textstyle\sum_{j=1}^N a_{j0} ||\tilde{\mathbf{v}}_j^a - \tilde{\mathbf{v}}_0^a||^2, \end{aligned}$$

where $\tilde{\mathbf{v}}_j^a$ is defined in (4) and the center of the formation is a point on $\tilde{\mathbf{v}}_0^a$. Q_{pitch} is minimum when the pitch of every particle is equal to α_0. The parameter $a_{j0} = 1$ if information flows from particle 0 to particle $j \in \mathcal{N}$, and $a_{j0} = 0$ otherwise. Consider the augmented potential

$$Q_0(\tilde{\mathbf{v}}^a) = Q(\tilde{\mathbf{v}}^a) + Q_{pitch}(\tilde{\mathbf{v}}^a) + Q_{center}(\tilde{\mathbf{v}}^a), \tag{17}$$

which is non-increasing under the control

$$\begin{aligned} \tilde{\mathbf{u}}_k = \; & K_0 \tilde{R}_k^T [s_k(t)\boldsymbol{\omega}_0 + (\tilde{\mathcal{L}}_k \tilde{\mathbf{v}}^a + a_{k0}(\beta_k \boldsymbol{\omega}_0 / \|\boldsymbol{\omega}_0\| \\ & + \tilde{\mathbf{v}}_k^a - \tilde{\mathbf{v}}_0^a)) \times \tilde{\mathbf{x}}_k], \;\; K_0 > 0. \end{aligned} \tag{18}$$

Fig. 1(b) illustrates (18) stabilizing a helical formation with $\alpha_0 = 0.8$ and the center located at the origin of the xy-plane.

Corollary 2: All solutions of the closed-loop model (2) with time-varying flow $\mathbf{f}_k(t)$, the speed $s_k(t)$ given by (5), and the control $\tilde{\mathbf{u}}_k$ by (18), converge to the set $\{\dot{Q}_0 = 0\}$, where Q_0 is defined in (17). The set of helical formations centered on $\tilde{\mathbf{v}}_0^a$, with pitch α_0, and radius $\|\boldsymbol{\omega}_0\|^{-1}$ is uniformly asymptotically stable and the formation center is determined by the initial conditions.

V. Motion Coordination on a Rotating Sphere

The spherical model is a special case of the three-dimensional model in which the particles travel on the surface of a (rotating) sphere in the presence of a spatiotemporal flowfield. We seek to design a decentralized control law that stabilizes circular formations on the sphere, as has been done previously for the flow-free case [7]. We first provide a control law to stabilize a circular formation on a rotating sphere and then a control law to stabilize a circular formation on a rotating sphere in a spatiotemporal flowfield. The potential [7]

$$V(\mathbf{r}, \mathbf{x}, \mathbf{y}, \mathbf{z}) \triangleq \tfrac{1}{2}\mathbf{c}^T \mathcal{L} \mathbf{c} = \tfrac{1}{2} \textstyle\sum_{(j,k) \in E} \|\mathbf{c}_j - \mathbf{c}_k\|^2, \tag{19}$$

where $\mathcal{L} \triangleq L \otimes I_3$ and $\mathbf{c}_k = \mathbf{r}_k + \omega_0^{-1} \mathbf{y}_k$, is minimized by the set of circular formations on a sphere in accordance with Lemma 3. The time derivative of V along solutions of (8) is $\dot{V} = \sum_{j=1}^N \dot{\mathbf{c}}_j \cdot \mathcal{L}_j \mathbf{c} = \sum_{j=1}^N (1 - \omega_0^{-1} \nu_j) \mathbf{x}_j \cdot \mathcal{L}_j \mathbf{c}$. Choosing

$$\nu_k = \omega_0 (1 + K_0 \mathbf{x}_k \cdot \mathcal{L}_k \mathbf{c}), \; K_0 > 0, \tag{20}$$

yields $\dot{V} = -K_0 \sum_{j=1}^N (\mathbf{x}_j \cdot \mathcal{L}_j \mathbf{c})^2 \leq 0$, which ensures that V is nonincreasing. Using (7) to compute $u_k = \nu_k + 2\omega_1 z_{k_3}$ we cancel the Coriolis effect and stabilize a circular formation with a fixed center. The following theorem extends [7, Theorem 4] to motion on a rotating sphere.

Theorem 4: All solutions of the closed-loop (flow-free) model (8), where the control ν_k is given by (20), converge to the set $\{\dot{V} = 0\}$, where V is defined in (19). The set of circular formations with radius $|\omega_0|^{-1}$ and direction of rotation determined by the sign of ω_0 is asymptotically stable.

Proof: By the invariance principle, all solutions converge to the largest invariant set where $\mathbf{x}_k \cdot (\mathbf{c}_k - \mathbf{c}_j) = 0$ for all connected pairs j and k. This is a set of circular trajectories on the same or opposite sides of the sphere. The set of circular formations (on the same side of the sphere) is asymptotically stable because it corresponds to the global minimum V. ■

Next we incorporate a spatiotemporal flow to the rotating spherical model. We work in the frame $\mathcal{D}_k$ to find a control law that stabilizes a circular formation. The potential

$$\tilde{V}(\mathbf{r}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}, \tilde{\mathbf{z}}) = \frac{1}{2}\tilde{\mathbf{c}}^T \mathcal{L}\tilde{\mathbf{c}} = \frac{1}{2}\sum_{(j,k)\in E}^{N} \|\tilde{\mathbf{c}}_j - \tilde{\mathbf{c}}_k\|^2, \quad (21)$$

where $\mathcal{L} \triangleq L \otimes I_3$ and $\tilde{\mathbf{c}}_k$ is defined in (10), is minimized by the set of circular formations on the sphere in accordance with Lemma 4. The time derivative of $\tilde{V}$ along solutions of (9) is $\dot{\tilde{V}} = \sum_{j=1}^{N} \dot{\tilde{\mathbf{c}}}_j \cdot \mathcal{L}_j \tilde{\mathbf{c}} = \sum_{j=1}^{N} (s_j(t) - \omega_0^{-1}\tilde{\nu}_j)\tilde{\mathbf{x}}_j \cdot \mathcal{L}_j \tilde{\mathbf{c}}$. Choosing the control law

$$\tilde{\nu}_k = \omega_0(s_k(t) + K_0 \tilde{\mathbf{x}}_k \cdot \mathcal{L}_k \tilde{\mathbf{c}}),\ K_0 > 0, \quad (22)$$

ensures $\dot{\tilde{V}} = -K_0 \sum_{j=1}^{N} (\tilde{\mathbf{x}}_j \cdot \mathcal{L}_j \tilde{\mathbf{c}})^2 \leq 0$.

Theorem 5: All solutions of the closed-loop model (9), with time-varying flow $\mathbf{f}_k(t)$, the speed $s_k(t)$ given by (5), and the control $\tilde{\nu}_k$ by (22) converge to the set $\{\dot{\tilde{V}} = 0\}$, where $\tilde{V}$ is defined in (21). The set of circular formations with radius $|\omega_0|^{-1}$ and direction of rotation determined by the sign of ω_0 is uniformly asymptotically stable.

Proof: The closed-loop dynamics (9) with $\tilde{\nu}_k$ given in (22) depend on the time-varying speed $s_k(t)$. Therefore, the proof follows from application of a pair of invariance-like theorems [19, Theorems 8.4 and 8.5]. The potential $\tilde{V}$ is radially unbounded and positive-definite in the reduced space of relative centers. The time derivative of $\tilde{V}$ satisfies $\dot{\tilde{V}} \leq 0$ and neither $\tilde{V}$ nor $\dot{\tilde{V}}$ depend explicitly on time. By [19, Theorem 8.4] all solutions converge to the set $\{\dot{\tilde{V}} = 0\}$, in which $\tilde{\mathbf{x}}_k \cdot \mathcal{L}_k \tilde{\mathbf{c}} = 0\ \forall\ k$, which includes circular trajectories on the same or opposite sides of the sphere. Since $\tilde{V}$ is minimized by the set of circular formations for which $\tilde{\mathbf{c}}_k = \tilde{\mathbf{c}}_j\ \forall k, j \in \mathcal{N}$, this set is asymptotically stable, uniformly in time. ■

Fig. 1(c) illustrates the stabilization of a circular formation in a spatiotemporal flow generated by two point vortices on the sphere. The flowfield at $\mathbf{r}_k$ due to M point vortices of identical strength Γ is $\mathbf{f}_k(t) = \frac{1}{4\pi\rho_0}\sum_{j=1;j\neq k}^{M} \Gamma_j \frac{\mathbf{r}_j \times \mathbf{r}_k}{\rho_0^2 - \mathbf{r}_k \cdot \mathbf{r}_j}$ [20], where $\mathbf{r}_j$ in the sum is the position of the jth vortex.

Under control (22) the center of the circular formation depends on the initial conditions of the particles. In order to prescribe the formation center, we introduce a virtual particle $k = 0$ with dynamics given by (9) with $\nu_0 = \omega_0 s_0(t)$, so that $\tilde{\mathbf{c}}_0(t) = \tilde{\mathbf{c}}_0(0)\ \forall\ t$. Consider the augmented potential

$$\tilde{V}_0(\mathbf{r}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}, \tilde{\mathbf{z}}) = \frac{1}{2}\tilde{\mathbf{c}}^T \mathcal{L}\tilde{\mathbf{c}} + \frac{1}{2}\sum_{j=1}^{N} a_{j0}||\tilde{\mathbf{c}}_j - \tilde{\mathbf{c}}_0||^2, \quad (23)$$

where $a_{j0} = 1$ if there is information flow from particle 0 to particle $j \in \mathcal{N}$, and $a_{j0} = 0$ otherwise. We ensure $\tilde{V}_0$ is non-increasing by choosing

$$\tilde{\nu}_k = \omega_0 \left(s_k(t) + K_0 \tilde{\mathbf{x}}_k \cdot \left[\mathcal{L}_k \tilde{\mathbf{c}} + a_{k0}\left(\tilde{\mathbf{c}}_k - \tilde{\mathbf{c}}_0\right)\right]\right). \quad (24)$$

Control (24) stabilizes the set of circular formations on a rotating sphere with prescribed center c_0.

VI. Conclusion

In this note we present a Lyapunov-based design of decentralized control algorithms for a three-dimensional, connected network of self-propelled particles in a spatiotemporal flowfield. We provide control laws to stabilize parallel formations with arbitrary and prescribed direction in three dimensions. We also provide control laws to stabilize helical formations in three dimensions with arbitrary pitch and center and prescribed pitch and center. In a spherical model that is a special case of the three-dimensional model, we provide control laws to stabilize circular formations on a rotating sphere with arbitrary and prescribed center. In ongoing work we seek decentralized control algorithms to regulate particle space-time separation in a three-dimensional, spatiotemporal flowfield.

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