Stabilization of Symmetric Formations to Motion around Convex Loops

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Abstract

We provide a cooperative control algorithm to stabilize symmetric formations to motion around closed curves suitable for mobile sensor networks. This work extends previous results for stabilization of symmetric circular formations. We study a planar particle model with decentralized steering control subject to limited communication. Because of their unique spectral properties, the Laplacian matrices of circulant graphs play a key role. We illustrate the result for a skewed superellipse, which is a type of curve that includes circles, ellipses, and rounded parallelograms.

Key words: cooperative control, curvature, sensor networks, oscillators, Laplacian

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1 Introduction

Cooperative control is an emerging field of study that has many applications including modeling biological aggregations and designing mobile sensor networks. In particular, coordinated sensing with autonomous underwater vehicles (AUV) motivates our work, although the main result is sufficiently decoupled from this application to justify its use in other settings. We describe a multiple vehicle control algorithm to stabilize symmetric formations on a parameterized family of trajectories suitable for oceanographic sampling. Symmetric formations, in which vehicles are uniformly arranged on one or more closed paths, minimize the collective mapping error of a dynamic ocean process like temperature or salinity [1]. Oceanographers prefer closed paths with long, nearly straight sides because they can interpret the sensor measurements collected along repeated orbits without using a complex ocean model. We treat the design of collective trajectories for ocean sampling as a decentralized problem and use cooperative control laws subject to limited communication.

This paper utilizes and contributes to the literature on cooperative control algorithms. We model each autonomous vehicle as a Newtonian point mass (particle) constrained to a plane and subject to a gyroscopic steering force, after [2]; this is a second-order, under-actuated, and constant speed particle model. Particle models with limited communication have received much attention; there is a large literature on multi-agent consensus, agreement, and rendezvous on Euclidean spaces [3,4,5,6]. We study synchronization, which is consensus on the torus, and a complementary notion that we call balancing. Our framework enables synchronization and balancing of particle formations using curvature and arc length separation as feedback in a coupled oscillator model; this approach differs from other multi-vehicle control algorithms for boundary tracking [7,8] and pattern formation [9].

This paper extends our previous results for stabilization of symmetric circular formations with all-to-all communication [10] and limited communication [11]. The design and demonstration of a real-time, multiple AUV control framework that implements the algorithms here are described in [12]. The main contribution of this paper is a cooperative control law to stabilize symmetric formations on convex, closed curves (convex loops). In Section 2, we describe the motion and communication models as well as introduce our curve framework. In Section 3, we present algorithms to (i) control particles to the same curve; (ii) control relative spacing along the curve; and (iii) control to formations on the same curve with symmetric relative spacings.

2 Model descriptions

Particle and phase models We model N Newtonian point masses (particles) subject to the constraint that the positions $r_k = x_k + iy_k \in \mathbb{C}$, $k = 1, \ldots, N$, and velocities $\dot{r}_k \in \mathbb{R}^2 \equiv \mathbb{C}$. The force acting on each particle is orthogonal to its velocity, which implies that the particles travel at constant speed. If the initial speed of each particle is equal to one, then the velocity of the kth particle is the unit phasor $e^{i\theta_k}$, where $\theta_k \in \mathbb{T}$ is the phase angle that describes the direction of motion of the kth particle. We denote by $\mathbb{T}$ the one-torus, that is $\theta_k \in \mathbb{T}$ implies we identify $\theta_k + 2\pi$ with θ_k. The steering control $u_k \in \mathbb{R}$ is the magnitude of the force on the kth particle.

We adopt the following notation. We drop the subscript and use bold to represent a vector of length N such as $\mathbf{r} \triangleq (r_1, \ldots, r_N)^T \in \mathbb{C}^N$ and $\boldsymbol{\theta} \triangleq (\theta_1, \ldots, \theta_N)^T \in \mathbb{T}^N$. Analogously, $e^{i\boldsymbol{\theta}} \triangleq (e^{i\theta_1}, \ldots, e^{i\theta_N})^T \in \mathbb{C}^N$. For vectors $\mathbf{w} \in \mathbb{C}^N$ and $\mathbf{z} \in \mathbb{C}^N$, we use the inner product $\langle \mathbf{w}, \mathbf{z} \rangle \triangleq \mathrm{Re}\{\mathbf{w}^* \mathbf{z}\}$, where * denotes conjugate transpose; for $w \in \mathbb{C}$ and $z \in \mathbb{C}$ we write $\langle w, z \rangle \triangleq \mathrm{Re}\{w^* z\}$.

Next we write down the particle motion models and provide several simple examples. The *particle model* is [2]

$$\begin{aligned} \dot{r}_k &= e^{i\theta_k} \\ \dot{\theta}_k &= u_k(\mathbf{r}, \boldsymbol{\theta}),\ k = 1, \ldots, N. \end{aligned} \tag{1}$$

In our approach, we find it useful to also consider the *phase model* given by

$$\dot{\theta}_k = u_k(\boldsymbol{\theta}),\ k = 1, \ldots, N. \tag{2}$$

If the control $u_k = 0$, then phase θ_k is constant and particle k moves in a straight line. If the control $u_k = \omega_0 \neq 0$, then phasor $e^{i\theta_k}$ revolves around the unit circle and particle k travels around a circle with radius $|\omega_0|^{-1}$ and center $c_k \triangleq r_k + \omega_0^{-1} i e^{i\theta_k}$.

We formalize important terminology for phase arrangements $\boldsymbol{\theta}$. Let $\mathbf{1} \triangleq (1, \ldots, 1)^T \in \mathbb{R}^N$. If $\boldsymbol{\theta} = \theta_0 \mathbf{1}$, $\theta_0 \in \mathbb{T}$, then $\boldsymbol{\theta}$ is synchronized. In the phase model, synchronization implies all the phases are equal; in the particle model, phase synchronization implies all particles move in the same direction. If $\boldsymbol{\theta}$ satisfies $\mathbf{1}^T e^{i\boldsymbol{\theta}} = 0$, then $\boldsymbol{\theta}$ is balanced. In the phase model, balancing implies that the phasor centroid is zero; in the particle model, phase balancing implies that average particle velocity is zero and the center of mass is fixed. We refer to synchronization and balancing of $m\boldsymbol{\theta}$, $m \in \mathbb{N} \triangleq \{1, 2, 3, \ldots\}$, as synchronization and balancing modulo $\frac{2\pi}{m}$.

Communication model The communication between particles may be limited in the sense that each particle receives information from only some of the other particles. The implication of this limitation is that the control u_k is a function of the positions and velocities of the set of particles from which particle k receives information. We call this set the neighbors of k and denote it by $\mathcal{N}_k$. The collective communication topology is a graph, $G(t)$, which, in general, may be time-varying and directed. Each node or vertex in the graph corresponds to a particle; an edge from the jth to the kth node represents information flow from j to k.

We use two matrix representations of the graph $G(t)$, which are the incidence matrix $B(t)$ and Laplacian matrix $L(t)$. Assume that the edges of $G(t)$ all have unit weight. Let $e(t)$ denote the number of edges. The incidence matrix at time t has dimensions $N \times e(t)$. Each column of the incidence matrix corresponds to an edge; if the edge connects the j to the kth node, then the jth element of the column is -1, the kth element is 1 and all other elements are zero. The elements of the Laplacian matrix, which is a square $N \times N$ matrix, are: $[L]_{k,j} \triangleq -1$, $j \neq k$, if there is an edge from j to k at time t and zero otherwise; and $[L]_{k,k} \triangleq |\mathcal{N}_k|$, which is the number of neighbors of k.

Next, we describe several well known properties of the graph Laplacian that result from assumptions about the graph $G(t)$. If $G(t)$ is undirected, which means that j is a neighbor of k if and only if k is a neighbor of j, then $L(t) = L(t)^T = B(t)B(t)^T$. Secondly, if $G(t)$ is strongly connected, which means that there is a path along edges between any two distinct nodes that respects edge direction, then the kernel of $L(t)$ is spanned by $\mathbf{1}$, that is $L(t)\mathbf{x} = 0$ if and only if $\mathbf{x} = x_0\mathbf{1}$. Let $\bar{G}$ denote a time-invariant graph that is undirected and strongly connected. Graph $\bar{G}$ is complete if there is an edge between every pair of nodes. A complete graph is circulant; in general, $\bar{G}$ is circulant if its Laplacian matrix is a circulant matrix [13].

Curve-phase model We now derive the open-loop control u_k that drives particle k around a smooth, convex and closed curve C that has definite curvature. In this setting, the velocity of particle k is tangent to C. Let c_k denote the center of C. We parameterize C in a reference frame with origin c_k using $\rho : [0, 2\pi) \to \mathbb{C}$, $\phi \mapsto \rho(\phi)$, where $\phi : \mathbb{T} \to [0, 2\pi)$, $\theta_k \mapsto \phi(\theta_k)$, is a smooth map. The tangent vector to C is $\frac{d\rho}{d\phi}$, which implies the velocity constraint $\dot{r}_k = e^{i\theta_k} = \left|\frac{d\rho}{d\phi}\right|^{-1} \frac{d\rho}{d\phi}$.

Next, we consider the curvature of C. The arc length along the curve is

$$\sigma(\phi) \triangleq \int_0^{\phi} \left|\frac{d\rho}{d\bar{\phi}}\right| d\bar{\phi}. \tag{3}$$

Under the velocity constraint, the local curvature of C is

$$\kappa(\phi) \triangleq \pm \frac{d\theta_k}{d\sigma}, \tag{4}$$

where the sign determines the sense of rotation. By assumption, the curvature of C is bounded and definite, that is $0 < |\kappa(\phi)| < \infty$. Using (3) and (4),

$$\kappa^{-1}(\phi) = \frac{1}{\kappa(\phi)} = \pm \frac{d\sigma}{d\theta_k} = \pm \frac{d\sigma}{d\phi}\frac{d\phi}{d\theta_k} = \pm \left|\frac{d\rho}{d\phi}\right| \frac{d\phi}{d\theta_k}. \tag{5}$$

Consequently, using (5) and the velocity constraint,

$$\frac{d\rho}{d\theta_k} = \frac{d\rho}{d\phi}\frac{d\phi}{d\theta_k} = \pm e^{i\theta_k}\kappa^{-1}(\phi). \tag{6}$$

The curve-phase ψ is the angular displacement of a point along the curve,

$$\psi(\phi) \triangleq \frac{2\pi}{\Omega}\sigma(\phi), \tag{7}$$

where $\Omega = \sigma(2\pi) > 0$ is the perimeter [9]. The definitions for synchronization and balancing of phase arrangement $\boldsymbol{\theta}$ apply to the curve-phase arrangement $\boldsymbol{\psi} \triangleq (\psi_1, \ldots, \psi_N)^T \in \mathbb{T}^N$. Using (5) and (7), we obtain

$$\frac{d\psi}{dt} = \frac{2\pi}{\Omega}\frac{d\sigma}{d\theta_k}\frac{d\theta_k}{dt} = \pm\frac{2\pi}{\Omega}\kappa^{-1}(\phi)\dot{\theta}_k. \tag{8}$$

Let $\rho_k \triangleq \rho(\phi(\theta_k))$, $\kappa_k \triangleq \kappa(\phi(\theta_k))$, and $\psi_k \triangleq \psi(\phi(\theta_k))$ for $k \in \{1, \ldots, N\}$. By analogy to the phase model (2), the *curve-phase model* is

$$\dot{\psi}_k = \pm\frac{2\pi}{\Omega}\kappa_k^{-1}u_k(\boldsymbol{\psi}). \tag{9}$$

The center of C is $c_k \triangleq r_k \mp \rho_k$. The definition for the center of circular motion of radius $|\omega_0|^{-1}$ is a special case with $\rho_k = -i\omega_0^{-1}e^{i\theta_k}$. Using (1) and (6), the center velocity is

$$\dot{c}_k = e^{i\theta_k}(1 - \kappa_k^{-1}u_k). \tag{10}$$

Using equations (10) and (9), we obtain an expression for the open-loop control that drives particle k around curve C with a fixed center. The curve control is $u_k = \kappa_k$, which implies $\dot{c}_k = 0$ and $\dot{\psi}_k = \pm\frac{2\pi}{\Omega}$. We illustrate the curve-phase and center of an ellipse in Figure 1.

As a more general example, we consider the (skewed) superellipse, a class of curves that includes circles, ellipses, and rounded parallelograms. We have

$$\rho(\phi) = a(\cos\phi)^{\frac{1}{p}} + (i + \mu)b(\sin\phi)^{\frac{1}{p}} \tag{11}$$

where $\mu \in \mathbb{R}$ is the skew parameter. The semi-major axis length a and semi-minor axis length b satisfy $a \geq b > 0$. The parameter $p = 1, 3, 5, \ldots$ determines the corner sharpness. For $\mu = 0$ and $a > b$ (resp. $a = b$), setting $p = 1$ yields an ellipse (resp. circle) and setting $p \geq 3$ yields a rounded rectangle (resp.

rounded square). Setting $\mu \neq 0$ and $p > 1$ yields a rounded parallelogram. In Appendix A, we solve for the parametrization ρ in (A.2) and the curvature κ in (A.1). For example, setting $\mu = 0$ and $p = 1$ yields the ellipse curvature

$$\kappa_k = \frac{1}{a^2 b^2} \left(a^2 \sin^2 \theta_k + b^2 \cos^2 \theta_k \right)^{\frac{3}{2}}. \tag{12}$$

Setting $b = a$ in (12) yields the constant curvature of a circle with radius a, $\kappa_k = \frac{1}{a}$. The curves and curvatures for $p = 1$ and 3 are illustrated in Figure 2 for $\mu = 0$ and positive ($\kappa_k > 0$) rotation. Without loss of generality, we assume $\kappa_k > 0$ in the rest of the paper.

3 Control of particle formations

In this section, we generalize previous results for stabilization of symmetric circular formations of the particle model (1) with all-to-all [10] and limited communication [11]. In the current setting, we provide feedback controls to stabilize symmetric formations on curve C. We present here the case when the communication graph $\bar{G}$ is fixed, undirected, and strongly connected; this case and its extension to time-varying and directed graphs build on [11].

We derive stabilizing controls by designing Lyapunov functions that are minimum in the desired configuration. Let L be the Laplacian of $\bar{G}$ and $\mathbf{z} \in \mathbb{C}^N$. The Laplacian quadratic form $Q(\mathbf{z}) \triangleq \frac{1}{2}\langle \mathbf{z}, L\mathbf{z} \rangle$ is zero for $\mathbf{z} = z_0 \mathbf{1}$, $z_0 \in \mathbb{C}$, and positive otherwise; thus, $Q(e^{i\boldsymbol{\psi}}) = 0$ if and only if $\boldsymbol{\psi}$ is synchronized.

Curve control In order to drive the particles to curve C with a common center, we choose a stabilizing control that minimizes the Laplacian quadratic form

$$S(\mathbf{r}, \boldsymbol{\theta}) \triangleq Q(\mathbf{c}) = \frac{1}{2}\langle \mathbf{c}, L\mathbf{c} \rangle. \tag{13}$$

Note, $S(\mathbf{r}, \boldsymbol{\theta})$ is zero if all centers coincide, that is $\mathbf{c} = c_0\mathbf{1}$, $c_0 \in \mathbb{C}$, and is positive otherwise. The time-derivative of $S(\mathbf{r}, \boldsymbol{\theta})$ along the solutions of (1) is $\dot{S}(\mathbf{r}, \boldsymbol{\theta}) = \sum_{k=1}^{N} \langle e^{i\theta_k}, L_k\mathbf{c} \rangle (1 - \kappa_k^{-1} u_k)$, where L_k denotes the kth row of L. Choosing the Laplacian curve control

$$u_k = \kappa_k(1 + K_0\langle e^{i\theta_k}, L_k\mathbf{c} \rangle), \;\; K_0 > 0 \tag{14}$$

results in $\dot{S}(\mathbf{r}, \boldsymbol{\theta}) = -K_0 \sum_{k=1}^{N} \langle e^{i\theta_k}, L_k\mathbf{c} \rangle^2 \leq 0$. The closed-loop particle model with control (14) depends only on the particle (absolute) phases and the relative position of the curve centers of neighboring particles. Therefore, the closed-loop system is invariant to rigid translation of all curve centers. Lyapunov analysis provides the following.

Theorem 1 *All solutions of the particle model (1) with Laplacian curve control (14) converge to the set in which each particle orbits curve C with a common center.*

Proof: The function $S(\mathbf{r}, \boldsymbol{\theta})$ is positive definite and proper in the co-dimension 2 reduced space of the N particle phases θ_k and the $N-1$ relative positions of the curve centers c_k. Since $S(\mathbf{r}, \boldsymbol{\theta})$ is nonincreasing, by the LaSalle Invariance principle, solutions in the reduced space converge to the largest invariant set where $\langle e^{i\theta_k}, L_k\mathbf{c} \rangle \equiv 0$ for $k = 1, \ldots, N$. In this set, $\dot{\theta}_k = \kappa_k$ and c_k is constant, which means the invariance condition holds only if $L\mathbf{c} \equiv 0$, that is $\mathbf{c} = c_0\mathbf{1}$, $c_0 \in \mathbb{C}$. We conclude each particle orbits curve C with a common center c_0 that depends only on initial conditions. $\square$

Curve-phase control To control the (relative) curve-phase, we use the Laplacian quadratic form

$$W_m(\boldsymbol{\psi}) \triangleq Q(\frac{1}{m}e^{im\boldsymbol{\psi}}) = \frac{1}{2m^2}\langle e^{im\boldsymbol{\psi}}, Le^{im\boldsymbol{\psi}}\rangle. \tag{15}$$

Note, $W_m(\boldsymbol{\psi})$ is zero for $\boldsymbol{\psi}$ synchronized modulo $\frac{2\pi}{m}$ and positive otherwise. The gradient of $W_m(\boldsymbol{\psi})$ is $\frac{\partial W_m}{\partial \psi_k} = \frac{1}{m}\langle ie^{im\psi_k}, L_k e^{im\boldsymbol{\psi}}\rangle$, $k = 1, \ldots, N$. In the curve-phase model (9), choosing the Laplacian curve-phase control

$$u_k = K_m \kappa_k \frac{\partial W_m}{\partial \psi_k}, \quad K_m \neq 0 \tag{16}$$

yields $\dot{W}_m(\boldsymbol{\psi}) = \frac{K_m}{m}\frac{2\pi}{\Omega}\sum_{k=1}^{N}\langle ie^{im\psi_k}, L_k e^{im\boldsymbol{\psi}}\rangle^2$. The closed-loop curve-phase model with control (16) depends only the relative curve-phases of neighboring particles. Therefore, the closed-loop curve-phase model is invariant to rigid rotation of all curve-phases. Lyapunov analysis provides the following.

Theorem 2 *All solutions of the curve-phase model (9) with Laplacian curve-phase control (16) converge to the set of critical points of $W_m(\boldsymbol{\psi})$. For $K_m < 0$, the set of curve-phase arrangements that are synchronized modulo $\frac{2\pi}{m}$ is locally exponentially stable.*

Proof: The function $W_m(\boldsymbol{\psi})$ is positive definite in the compact, $N-1$ dimensional space of relative curve-phases. The evolution of $W_m(\boldsymbol{\psi})$ is monotonic along the solutions of (9). In particular, $W_m(\boldsymbol{\psi})$ is nonincreasing (nondecreasing) for $K_m < 0$ ($K_m > 0$). Using the LaSalle Invariance principle, solutions converge to the largest invariant set where $< ie^{im\psi_k}, L_k e^{im\boldsymbol{\psi}} > \equiv 0$ for $k = 1, \ldots, N$, which is the set of critical points of $W_m(\boldsymbol{\psi})$. The set of curve-phase arrangements that are synchronized modulo $\frac{2\pi}{m}$ is an isolated global minimum of $W_m(\boldsymbol{\psi})$ in the reduced space of relative curve-phases; as a result, this set is asymptotically stable for $K_m < 0$. Exponential stability of the set follows from linearization of the closed-loop curve phase model about $m\boldsymbol{\psi} = \psi_0\mathbf{1}$, $\psi_0 \in \mathbb{T}$. $\square$

If $\bar{G}$ is complete, then the set of curve-phase arrangements that are balanced modulo $\frac{2\pi}{m}$ is a global maximum of $W_m(\boldsymbol{\psi})$ in the reduced space of relative curve-phases; this set is asymptotically stable for $K_m > 0$ [10]. If $\bar{G}$ is not complete, a sufficient condition for $W_m(\boldsymbol{\psi})$ to have critical points other than synchronization is if $\bar{G}$ is a circulant graph, which implies L is a circulant matrix. All circulant matrices are diagonalized by the unitary discrete Fourier transform matrix [13], which yields the following lemma.

Lemma 3 *[11] Let $\bar{G}$ be a circulant graph. Set $\phi_k \triangleq (k-1)\frac{2\pi}{N}$ for $k = 1,\ldots,N$. Then the vectors $\mathbf{f}^{(l)} \triangleq e^{i(l-1)\boldsymbol{\phi}}$, $l = 1,\ldots,N$, define a basis of N orthogonal eigenvectors of the Laplacian L. The unitary matrix F whose columns are the N (normalized) eigenvectors $\frac{1}{\sqrt{N}}\mathbf{f}^{(l)}$ diagonalizes L, that is, $L = F\Lambda F^*$, where $\Lambda \triangleq \mathrm{diag}\{0,\lambda_2,\ldots,\lambda_N\} \geq 0$ is the eigenvalue matrix of L.*

Using the notation of Lemma 3, let $\boldsymbol{\psi}^{(l)} = (l-1)\boldsymbol{\phi}$, $l \in 1,\ldots,N$. If $\bar{G}$ is circulant, then $e^{i\boldsymbol{\psi}^{(l)}}$ is an eigenvector of the Laplacian L, which implies $\boldsymbol{\psi}^{(l)}$ is a critical point of the Laplacian quadratic form $W_m(\boldsymbol{\psi}^{(l)})$. If $l = 1$, then $\boldsymbol{\psi}^{(1)} = \mathbf{0} \triangleq (0,\ldots,0)^T \in \mathbb{R}^N$, which implies $\boldsymbol{\psi}^{(1)}$ is synchronized and $\mathbf{f}^{(1)} = e^{i\boldsymbol{\psi}^{(1)}} = \mathbf{1}$. If $l \neq 1$, by orthogonality of the $\mathbf{f}^{(l)}$, then $\mathbf{1}^T e^{i\boldsymbol{\psi}^{(l)}} = 0$, which implies $\boldsymbol{\psi}^{(l)}$ is balanced. In fact, $\boldsymbol{\psi}^{(l)}$ is a symmetric curve-phase pattern.

Symmetric formations In this section, we combine the results from Sections 3 and 3 to stabilize symmetric formations on curve C. Particles in a symmetric formation have a curve-phase arrangement that is a symmetric pattern. An (M,N)-pattern is a symmetric (curve-)phase arrangement that has M clusters of $\frac{N}{M}$ synchronized phases, where M is a divisor of N [10]. For any $N \geq 2$ phases, it is always possible to form at least two (M, N)-patterns: the synchronized $(1, N)$-pattern and the splay (N, N)-pattern. The splay pattern is an arrangement in which the phasors are uniformly distributed around the unit circle separated by multiples of $\frac{2\pi}{N}$. Note, if $e^{i\boldsymbol{\psi}}$ is a (M, N)-pattern then $\boldsymbol{\psi}$ is synchronized modulo $\frac{2\pi}{M}$ and balanced modulo $\frac{2\pi}{m}$, $m = 1, \ldots, M-1$.

To isolate symmetric curve-phase arrangements, we consider the sum of Laplacian quadratic forms

$$W_L^{M,N}(\boldsymbol{\psi}) \triangleq -\sum_{m=1}^{N} \frac{K_m}{m} W_m(\boldsymbol{\psi}) \tag{17}$$

where $W_m(\boldsymbol{\psi})$ is defined by (15). The gains must satisfy $K_m > 0$, $m = 1, \ldots, M-1$, and $K_M < -M \sum_{m=1}^{M-1} \frac{K_m}{m}$. If $\bar{G}$ is complete, then the set of curve-phase arrangements that are (M, N)-patterns is an isolated global minimum of $W_L^{M,N}(\boldsymbol{\psi})$ in the reduced space of relative curve-phases [10]. In the case of limited communication, we have the corresponding local result, adapted from [11]. The proof of Lemma 4 appears in Appendix B.

Lemma 4 *If $\bar{G}$ is circulant, then the set of curve-phase arrangements that are (M, N)-patterns is an isolated local minimum of $W_L^{M,N}(\boldsymbol{\psi})$ in the reduced space of relative curve-phases.*

In order to drive the particles to a symmetric formation on curve C, we choose a stabilizing control that minimizes the composite Lyapunov function

$$V(\boldsymbol{r}, \boldsymbol{\theta}) \triangleq K_0 S(\boldsymbol{r}, \boldsymbol{\theta}) + \frac{\Omega}{2\pi} W_L^{M,N}(\boldsymbol{\psi}) \tag{18}$$

where $S(\boldsymbol{r}, \boldsymbol{\theta})$ is defined by (13) and $W_L^{M,N}(\boldsymbol{\psi})$ is defined by (17). The time-derivative of $V(\boldsymbol{r}, \boldsymbol{\theta})$ along the solutions of (1) is

$$
\begin{aligned}
\dot{V}(\boldsymbol{r},\boldsymbol{\theta}) &= \sum_{k=1}^{N}\left(K_0\langle e^{i\theta_k}, L_k\boldsymbol{c}\rangle(1-\kappa_k^{-1}u_k) + \kappa_k^{-1}u_k\frac{\partial W_L^{M,N}}{\partial\psi_k}\right)\\
&= \sum_{k=1}^{N}\left(K_0\langle e^{i\theta_k}, L_k\boldsymbol{c}\rangle - \frac{\partial W_L^{M,N}}{\partial\psi_k}\right)(1-\kappa_k^{-1}u_k)
\end{aligned}
$$

where we used $\sum_{k=1}^{N}\frac{\partial W_m}{\partial\psi_k} = \langle ie^{im\boldsymbol{\psi}}, BB^Te^{im\boldsymbol{\psi}}\rangle = \langle iB^Te^{im\boldsymbol{\psi}}, B^Te^{im\boldsymbol{\psi}}\rangle = 0$. Choosing the Laplacian formation control

$$
u_k = \kappa_k\left(1 + K_0\langle e^{i\theta_k}, L_k\boldsymbol{c}\rangle - \frac{\partial W_L^{M,N}}{\partial\psi_k}\right),\ K_0 > 0 \tag{19}
$$

yields $\dot{V}(\boldsymbol{r},\boldsymbol{\theta}) = \sum_{k=1}^{N}\left(K_0\langle e^{i\theta_k}, L_k\boldsymbol{c}\rangle - \frac{\partial W_L^{M,N}}{\partial\psi_k}\right)^2 \leq 0$. The closed-loop particle model with control (19) is invariant to rigid translation of all the curve centers. Lyapunov analysis provides the following.

Theorem 5 *Let $\bar{G}$ be circulant and L be the corresponding Laplacian matrix. All solutions of the particle model (1) with Laplacian formation control (19) converge to the set in which (i) each particle orbits curve C with a common center and (ii) the curve-phase arrangement is a critical point of $W_L^{M,N}(\boldsymbol{\psi})$. The control locally exponentially stabilizes the set of curve-phase arrangements that are (M,N)-patterns.*

Proof: The proof is a straightforward adaptation of Theorem 7 in [11]. The function $V(\mathbf{r},\boldsymbol{\theta})$ is positive definite in the co-dimension 2 reduced space of the N particle phases θ_k and $N-1$ relative positions of the curve centers c_k. Since $V(\boldsymbol{r},\boldsymbol{\theta})$ is nonincreasing, by the LaSalle Invariance principle, solutions in the reduced space converge to the largest invariant set Λ where

$$
K_0\langle e^{i\theta_k}, L_k\mathbf{c}\rangle \equiv \frac{\partial W_L^{M,N}}{\partial\psi_k},\ k = 1,\ldots,N. \tag{20}
$$

In the set Λ, $\dot{\theta}_k = \kappa_k$ and $\dot{\psi}_k = \frac{2\pi}{\Omega}$, which implies that $W_L^{M,N}(\boldsymbol{\psi})$ is constant and $\dot{c}_k = 0$. Therefore, differentiating (20) with respect to time in Λ gives

$\langle ie^{i\theta_k}, L_k\mathbf{c}\rangle\kappa_k \equiv 0$ for $k = 1, \ldots, N$ which can hold only if $L\mathbf{c} \equiv 0$, i.e., all particles orbit curve C with a common center. Consequently, the invariance condition (20) becomes $\frac{\partial W_L^{M,N}}{\partial \psi_k} \equiv 0$ for $k = 1, \ldots, N$, which is satisfied by the set of critical points of $W_L^{M,N}(\boldsymbol{\psi})$. Therefore, the particles orbit curve C with common center c_0 in a curve-phase arrangement $\boldsymbol{\psi}$ that is a critical point of $W_L^{M,N}(\boldsymbol{\psi})$. By Lemma 4, the set of curve-phase arrangements that are (M, N)-patterns is exponentially stable because it is an isolated local minimum of $W_L^{M,N}(\boldsymbol{\psi})$ in the reduced space of relative curve-phases. $\square$

We further examine the Laplacian formation control through simulation. Theorem 5 does not exclude convergence to formations that correspond to other critical points of $W_L^{M,N}(\boldsymbol{\psi})$. Simulations indicate that the size of the basin of attraction depends on the connectivity of the communication graph. In fact, we don't observe convergence to a different critical point in simulations for a complete graph. Simulations also suggest that the magnitude of the gain $K_M < 0$ that is necessary to prove Lemma 4 is conservative. For example, simulations show local convergence to the desired symmetric pattern for all $|K_m|$ identical. We illustrate selected simulation results in Figure 3.

4 Conclusion

We present a cooperative control algorithm that stabilizes symmetric formations on a parameterized family of trajectories. The algorithm uses decentralized control of a planar particle model with limited communication. Particles converge to the set of trajectories that orbit a single closed curve. In our approach, we require the curve to be convex with definite curvature. Under this assumption, we exponentially stabilize symmetric patterns of relative curve-phase, which is the arc length spacing along the curve.

This work, which extends previous results for stabilizing symmetric circular formations, was motivated by the application to autonomous ocean sampling networks. Other application extensions, which are described in [11], include spatial symmetry breaking using reference particles and the design of the communication graph for multi-scale sampling. In ongoing work, we seek to develop cooperative control algorithms on non-planar surfaces, like the sphere, to support large-area ocean surveys.

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A Curvature and parameterization of skewed superellipse

Using (11), we have $\frac{d\rho}{d\phi} = -\frac{a}{p}(\cos\phi)^{\frac{1-p}{p}}\sin\phi + (i+\mu)\frac{b}{p}(\sin\phi)^{\frac{1-p}{p}}\cos\phi$ and, using the velocity constraint,

$$\tan\theta_k = \frac{\frac{b}{p}(\sin\phi)^{\frac{1-p}{p}}\cos\phi}{-\frac{a}{p}(\cos\phi)^{\frac{1-p}{p}}\sin\phi + \mu\frac{b}{p}(\sin\phi)^{\frac{1-p}{p}}\cos\phi}.$$

From these we compute $\cot\theta_k = -\frac{a}{b}(\cot\phi)^{\frac{1-p}{p}}\tan\phi + \mu$ and $\cot\phi = \left(\frac{b}{a}\left(\mu - \cot\theta_k\right)\right)^{\frac{p}{1-2p}}$. Consequently,

$$\frac{d\phi}{d\theta_k} = \frac{1+\cot^2\theta_k}{\frac{a}{b}\left(\frac{2p-1}{p}\right)\left(\frac{b}{a}(\mu-\cot\theta_k)\right)^{\frac{1-3p}{1-2p}}\left(1+\left(\frac{b}{a}(\mu-\cot\theta_k)\right)^{\frac{2p}{1-2p}}\right)}$$

and

$$\frac{d\rho}{d\phi} = \frac{\frac{a}{p}\left(\frac{b}{a}(\mu-\cot\theta_k)\right)^{\frac{p}{2p-1}}\left(-1+(i+\mu)(\mu-\cot\theta_k)^{-1}\right)}{\left(1+\left(\frac{b}{a}(\mu-\cot\theta_k)\right)^{\frac{2p}{2p-1}}\right)^{\frac{1}{2p}}}$$

Using $\frac{d\rho}{d\theta_k} = \frac{d\rho}{d\phi}\frac{d\phi}{d\theta_k}$, we obtain

$$\frac{d\rho}{d\theta_k} = \frac{ae^{i\theta_k}}{2p-1}\left[\sin\theta_k(\mu\sin\theta_k-\cos\theta_k)^2\left(1+\left(\frac{b}{a}(\mu-\cot\theta_k)\right)^{\frac{2p}{2p-1}}\right)^{\frac{1}{2p}}\times \left(1+\left(\frac{b}{a}(\mu-\cot\theta_k)\right)^{\frac{2p}{1-2p}}\right)\right]^{-1}$$

Using $\frac{d\rho}{d\theta_k} = e^{i\theta_k}\kappa_k^{-1}$, we obtain

$$\kappa_k = \frac{2p-1}{a}\sin\theta_k(\mu\sin\theta_k-\cos\theta_k)^2\left(1+\left(\frac{b}{a}(\mu-\cot\theta_k)\right)^{\frac{2p}{2p-1}}\right)^{\frac{1}{2p}}\times \left(1+\left(\frac{b}{a}(\mu-\cot\theta_k)\right)^{\frac{2p}{1-2p}}\right). \qquad \text{(A.1)}$$

We also find

$$\rho_k = \frac{a(\sin\theta)^{\frac{1}{2p-1}} + (i+\mu)b\left(\frac{b}{a}(\mu\sin\theta-\cos\theta)\right)^{\frac{1}{2p-1}}}{\left((\sin\theta)^{\frac{2p}{2p-1}} + \left(\frac{b}{a}(\mu\sin\theta-\cos\theta)\right)^{\frac{2p}{2p-1}}\right)^{\frac{1}{2p}}}. \tag{A.2}$$

B Proof of Lemma 4

Let $\bar{\boldsymbol{\psi}}$ be a (M,N)-pattern. First we prove that $\bar{\boldsymbol{\psi}}$ is a critical point of $W_L^{M,N}(\boldsymbol{\psi})$ [11, Lemma 2]. Modulo a uniform rotation and using the notation of Lemma 3, a (M,N)-pattern is characterized by the phase arrangement $\bar{\boldsymbol{\psi}} = \frac{N}{M}\boldsymbol{\phi}$. This means that the vector $e^{im\bar{\boldsymbol{\psi}}} = e^{im\frac{N}{M}\boldsymbol{\phi}}$ is the l-th eigenvector of L, with $l = 1 + \left(m\frac{N}{M}\right)$ mod N. But if $e^{im\bar{\boldsymbol{\psi}}}$ is an eigenvector of L, then $\bar{\boldsymbol{\psi}}$ is a critical point of $W_m(\boldsymbol{\psi})$, $m \in \{1,\ldots,M\}$.

Next we show that $\bar{\boldsymbol{\psi}}$ is a local minimum of $W_L^{M,N}(\boldsymbol{\psi})$ in the reduced space of relative curve-phases. We expand $W_L^{M,N}(\boldsymbol{\psi})$ about $\bar{\boldsymbol{\psi}}$ as

$$W_L^{M,N}(\bar{\boldsymbol{\psi}}+\delta\boldsymbol{\psi}) = W_L^{M,N}(\bar{\boldsymbol{\psi}}) + \delta\boldsymbol{\psi}^T H_L^{M,N}(\bar{\boldsymbol{\psi}})\delta\boldsymbol{\psi} + \mathcal{O}(\|\delta\boldsymbol{\psi}\|^3), \tag{B.1}$$

where $H_L^{M,N}(\boldsymbol{\psi})$ is the Hessian of $W_L^{M,N}(\boldsymbol{\psi})$. To find the Hessian, we evaluate

$$\frac{\partial^2 W_m}{\partial\psi_k^2} = \frac{1}{m}\sum_{j\in\mathcal{N}_k}\langle e^{im\psi_k}, e^{im\psi_j}\rangle \tag{B.2}$$

and, for $j \neq k$,

$$\frac{\partial^2 W_m}{\partial\psi_j\partial\psi_k} = \begin{cases} -\frac{1}{m}\langle e^{im\psi_k}, e^{im\psi_j}\rangle, & j\in\mathcal{N}_k, \\ \\ 0, & \text{otherwise.}\end{cases} \tag{B.3}$$

Evaluating (B.2) and (B.3) at $\bar{\boldsymbol{\psi}}$ yields the weighted Laplacian matrix $H_L^{M,N}(\bar{\boldsymbol{\psi}}) = -B\Phi_L^{M,N}(\bar{\boldsymbol{\psi}})B^T$ with the weight matrix $\Phi_L^{M,N}(\boldsymbol{\psi}) = \sum_{m=1}^{M}\frac{K_m}{m}\mathrm{diag}(\cos(mB^T\boldsymbol{\psi}))$.

Using $K_M < -M \sum_{m=1}^{M-1} \frac{K_m}{m}$, we have $\Phi_L^{M,N}(\bar{\boldsymbol{\psi}}) < 0$, which implies $H_L^{M,N}(\bar{\boldsymbol{\psi}}) \geq 0$. The zero eigenvalue is simple and corresponds to rigid rotation of all curve phases. Using (B.1), we have $W_L^{M,N}(\bar{\boldsymbol{\psi}} + \delta\boldsymbol{\psi}) > W_L^{M,N}(\bar{\boldsymbol{\psi}})$ in the reduced space, which completes the proof. □

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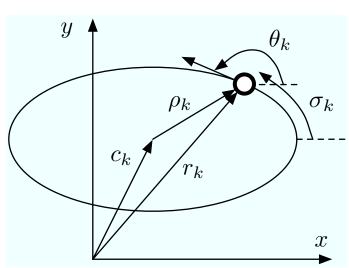

Fig. 1. The curve notation for the kth particle: the position and velocity direction of the particle are r_k and θ_k, respectively. The curve is centered at c_k.

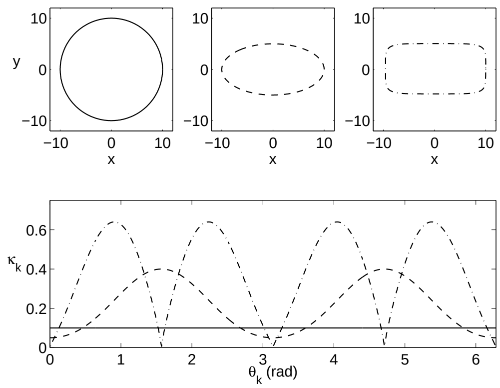

Fig. 2. *Top, left to right:* unskewed ($\mu = 0$) superellipses with $p = 1, a = b = 10$ (circle); $p = 1, a = 10, b = 5$ (ellipse); and $p = 3, a = 10, b = 5$ (rounded rectangle). *Bottom:* curvature κ_k as a function of tangent angle θ_k for a circle (solid), ellipse (dashed), and rounded rectangle (dash-dot). The open-loop control $u_k = \kappa_k$ drives particle k around the corresponding curve.

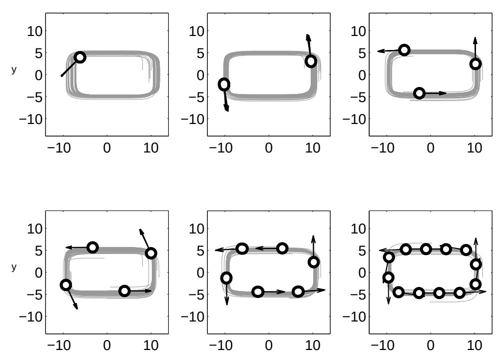

Fig. 3. Numerical simulations of the Laplacian formation control. Each particle is a black circle with an arrow to denote velocity; the trajectories are gray. We used $N = 12$, $K_0 = K_m = 0.1$, $m = 1, \ldots, M$, and random, local initial conditions. The communication graph $\bar{G}$ is circulant; each particle has four neighbors. The panels show six symmetric patterns on a superellipse with $\mu = 0$, $p = 3$ and $M = 1$ (synchronized), 2, 3, 4, 6, and 12 (splay pattern). The steady-state curve-phase differences between the clusters in each simulation are equal to $\frac{2\pi}{M}$.