Stabilization of Collective Motion of Self-Propelled Particles

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Summary. This paper presents analysis and design of feedback control laws for stabilization of parallel and circular trajectories of a network of self-propelled particles. Time-scale separation of inter-particle alignment and spacing controls permits application of previous convergence results for oscillator phase synchronization and particle motion.

1 Introduction

Collective motion occurs in large groups of natural organisms such as flocks of birds and schools of fish. One remarkable aspect of this motion is the stability of the aggregate linear momentum. Feedback control laws that stabilize group trajectories are important for engineering applications such as autonomous underwater vehicles (AUVs). For example, a fleet of AUVs can be used as a reconfigurable moving sensor array for data collection in the ocean [FBLS03]. We study a particle model that consists of self-propelled particles subject to planar steering controls, after [JK02]. The steering controls can be separated into relative alignment and spacing components. We use results from the coupled oscillator literature, namely [WS94], to stabilize the linear momentum of the group via relative orientation controls. The spacing controls are then used to achieve parallel and circular group motion.

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The particle model we consider in this paper is a kinematic model of self-propelled particles introduced in [JK02]. Each particle moves at constant speed in the plane but adapts its orientation (i.e. the direction of its unit velocity vector) according to the motion of neighboring particles. The state space of each particle is the group $SE(2) \approx \mathbb{R}^2 \times S^1$ of rigid displacements in the plane. The system of N coupled particles thus evolves on a high-dimensional state space (N copies of $SE(2)$). However, when the coupling only includes relative orientation variables and disregards relative spacing variables, the spatial variables can be ignored and the reduced dynamics evolve on N copies of S^1, i.e. an N-dimensional torus. The reduced dynamics then become equivalent to phase models of N coupled oscillators in which each oscillator is only modelled by a phase variable.

Analysis results have been obtained by Watanabe and Strogatz [WS94] for phase models of coupled oscillators under the particular assumption of (all-to-all) pure sinusoidal coupling. They showed that this model possesses $N-3$ constants of motion and provided a global analysis of the remaining low-dimensional dynamics. The oscillators either asymptotically synchronize or converge to a manifold of "incoherent" states. These incoherent states are characterized by an arbitrary distribution of phase differences on the unit circle such that the centroid of the oscillators vanishes. In our particle model, phase synchronization corresponds to parallel motion of the moving particles, whereas a vanishing centroid of the oscillators corresponds to motion of the particles around a fixed center of mass. Sinusoidal coupling of the orientation (phase) variables thus stabilizes the linear momentum of the group; either to a maximal value (parallel motion) or to a minimal value (fixed center of mass).

The connection between the phase-oscillator model of Watanabe and Strogatz [WS94] and the particle model of Justh and Krishnaprasad [JK02] is thus exploited to decouple the design of relative orientations from the design of relative spacing in particle models. The resulting dynamics have provable global convergence properties, and the proposed methodology should be applicable beyond the stabilization problems considered in this paper. The last section of the paper summarizes limitations of the present approach and possible directions for future research.

2 Particle Model

We consider a continuous-time kinematic model of N identical particles (of unit mass) moving in the plane at unit speed [JK02]:

$$\begin{aligned} \dot{r}_k &= e^{i\theta_k} \\ \dot{\theta}_k &= u_k, \quad 1 \leq k \leq N. \end{aligned} \tag{1}$$

In complex notation, the vector $r_k \in \mathbb{C} \approx \mathbb{R}^2$ denotes the position of particle k and the angle $\theta_k \in S^1$ denotes the orientation of its (unit) velocity vector

$e^{i\theta_k} = \cos\theta_k + i\sin\theta_k$. We omit the index of a coordinate to denote the corresponding N-vector, e.g. $r = (r_1, \dots, r_N)^T$. In the absence of steering control ($\dot{\theta}_k = 0$), each particle moves at unit speed in a fixed direction and its motion is decoupled from the other particles. We study the influence of various feedback control laws that result in coupled dynamics and closed-loop convergence to different types of *organized* or *collective* motion. We assume identical control for each particle. In that sense, the collective motions that we analyze in the present paper do not require differentiated control action for the different particles (e.g. the presence of a leader for the group). For convenience, we decompose the steering control of particle k into the sum of two terms:

$$u_k = u_k^{spac} + K u_k^{align} . \tag{2}$$

The term u_k^{align} depends only on relative orientation, i.e., on the variables $\theta_{jk} = \theta_j - \theta_k, 1 \leq j, k \leq N$. The term u_k^{spac} depends both on relative orientation and relative spacing, i.e., on the variables θ_{jk} and $r_{jk} = r_j - r_k, 1 \leq j, k \leq N$. The simplest form of phase coupling,

$$u_k^{align} = \frac{1}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_k), \tag{3}$$

is adopted throughout the paper, while different choices are discussed for the feedback u_k^{spac}. The sign of the parameter K plays an important role in the results of the paper.

The kinematic model (1) has been recently studied by Justh and Krishnaprasad [JK02, JK03]. These authors have emphasized the Lie group structure that underlies the state space. The configuration space consists of N copies of the group $SE(2)$. When the control law only depends on relative orientations and relative spacing, it is invariant under an action of the symmetry group $SE(2)$ and the closed-loop dynamics evolve on a reduced *shape space*. Equilibria of the reduced dynamics correspond to equilibrium shapes and can be only of two types [JK02]: *parallel* motions, characterized by a common orientation for all the particles, and *circular* motions, characterized by circular orbits of the particles around a fixed point. Both types of motion have been observed in simulations in a number of models that are kinematic or dynamic variants of the model (1), see for instance [LRC01].

A simplification of the model (1) occurs when the feedback laws depend on relative orientations only ($u_k^{spac} \equiv 0$). The control has then a much larger symmetry group (N copies of the translation group), and the reduced model becomes equivalent to phase models of the form

$$\dot{\theta}_k = \omega_k + \sum_j u_{jk}(\theta_j - \theta_k), \quad 1 \leq k \leq N \tag{4}$$

where the phase variable $(\theta_1, \dots, \theta_N)$ belongs to the N-dimensional torus T^N. (The model (4) still has an S^1-symmetry because the feedback only depends on phase differences). Formal equivalence with (1)-(2) is obtained with $u_k^{spac} = \omega_k$ and $K u_k^{align} = \sum_j u_{jk}(\theta_j - \theta_k)$. In the absence of coupling ($u_{jk} \equiv 0$ for all j and k), each particle "rotates" at its natural frequency ω_k around a fixed point. The phase coupling function, u_{jk}, between oscillator j and oscillator k is assumed to be continuously differentiable and 2π-periodic. The choice (3) assumes all-to-all coupling with an identical phase-difference coupling function that only includes one harmonic.

Our goal is to make the available results for phase models relevant to the particle model (1), even when the control u_k^{spac} is no longer constant. The proposed approach is to assume a large enough value for the parameter K such as to enforce a time-scale decomposition between the (fast) *orientation* dynamics (determined by the phase model (4)) and the (slow) *spacing* dynamics determined by the particle model (1) restricted to its slow manifold. Thus, we study the singularly perturbed model

$$\begin{aligned} \dot{r}_k &= e^{i\theta_k} \\ \epsilon\dot{\theta}_k &= \epsilon u_k^{spac} + \tfrac{1}{N}\textstyle\sum_{j=1}^{N} \sin(\theta_j - \theta_k), \quad 1 \le k \le N, \end{aligned} \tag{5}$$

where the small parameter ϵ enforces a time-scale separation between fast dynamics in the time-scale $\tau = \frac{t-t_0}{\epsilon}$ and slow dynamics in the time-scale t. In the fast time-scale τ, the variable r is frozen and the dynamics reduce to

$$\frac{d}{d\tau}\theta_k = \epsilon u_k^{spac} + \frac{1}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_k)$$

which, in the limit $\epsilon \to 0$, is precisely the phase model

$$\dot{\theta}_k = \frac{1}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_k), \quad 1 \le k \le N \tag{6}$$

studied by Strogatz and Watanabe. In the decomposition (2) of the feedback control, the control term u_k^{align} will determine the fast dynamics whereas the control term u_k^{spac} will affect the slow dynamics only.

We consider continuous-time models in this paper, but we mention a few relevant references to study their discrete-time counterpart. Couzin et al. [CKJ$^+$02] have studied such a model where the feedback control is determined from a set of simple rules: repulsion from close neighbors, attraction to distant neighbors, and preference for a common orientation. Their model includes stochastic effects but also exhibits collective motions reminiscent of either parallel motion or circular motion around a fixed center of mass. Interestingly, these authors have shown coexistence of these two types of motion in certain parameter ranges and hysteretic transition from one to the other. A discrete version of (4) has been studied by Viczek et al. [VCBJ$^+$95], in which ω_k is a random variable.

3 Linear Momentum Stabilization

The *centroid* of oscillators in the model (4) is defined by

$$p_\theta = \frac{1}{N}\sum_{k=1}^{N} e^{i\theta_k} (= \frac{1}{N}\sum_k \dot{r}_k) \tag{7}$$

Equivalently, p_θ is the linear momentum of the group of particles in the model (1). The codimension-two manifold defined by the algebraic condition $p_\theta = 0$ will be termed the *balanced* manifold, because individual motions of particles balance in this manifold to result in a fixed position for the center of mass of the group.[3]

Considering the singularly perturbed particle model (5), we can apply the conclusions of [SPL03] to the *fast* subsystem

$$(\text{ sign K})\dot{\theta}_k = \frac{1}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_k), \quad 1 \le k \le N. \tag{8}$$

Namely, the balanced manifold is globally attracting when $K < 0$ and the synchronized state is globally attracting when $K > 0$ [WS94].

If $K > 0$, (almost) all the solutions of (8) asymptotically synchronize. For the singularly perturbed model (5), the conclusion is that for K large enough, all solutions converge in the fast time scale, $\frac{t}{\epsilon}$, to an invariant slow manifold where the motion is nearly parallel. The slow manifold of parallel motion has codimension $N-1$; in the asymptotic limit $\epsilon \to 0$, it is determined by the $N-1$ algebraic conditions

$$\theta_k = \theta_1, \quad 1 < k \le N.$$

In contrast, if $K < 0$, then the solutions of (8) converge to the balanced manifold characterized by $p_\theta = 0$. For the singularly perturbed model (5), the conclusion is that for $|K|$ large enough, all solutions converge in the fast time scale, $\frac{t}{\epsilon}$, to an invariant slow manifold where the center of mass is nearly fixed. This slow manifold has codimension 2; in the asymptotic limit $\epsilon \to 0$, it is determined by the two algebraic conditions

$$\sum_{k=1}^{N} \cos\theta_k = \sum_{k=1}^{N} \sin\theta_k = 0\,.$$

In the next two sections, we illustrate how the slow dynamics can be designed to stabilize relative equilibria of (5).

[3] In the literature of phase oscillators, it has been termed the *incoherent state* manifold because $N-2$ phase differences are arbitrary in this manifold in contrast to the synchronized state in which all phase differences vanish.

4 Parallel Motion Design

To analyze the slow dynamics of the singularly perturbed particle model (5) in the case $\epsilon = 1/K > 0$, we determine the first-order approximation of its slow manifold. The slow manifold has the expression

$$\theta_k = \theta_1 + h_k(r, \epsilon), \quad 1 < k \leq N$$

To determine the functions h_k, $1 < k < N$, one expresses the invariance of the manifold

$$\theta_k - \theta_1 = h_k \Rightarrow \dot{\theta}_k - \dot{\theta}_1 = \dot{h}_k$$

As shown in [SPL03], the functions h_k have the first-order approximation

$$h_k(r, \epsilon) = \epsilon(u_k^{spac} - u_1^{spac}) + O(\epsilon^2), \;\; 1 < k \leq N\,. \tag{9}$$

The slow dynamics are obtained by substituting the approximation (9) in (5):

$$\dot{r}_k = e^{i\theta_1}(1 + i\epsilon(u_k^{spac} - u_1^{spac})) + O(\epsilon^2)\,. \tag{10}$$

For the difference $r_{jk} = r_j - r_k$, this yields the slow dynamics

$$\dot{r}_{jk} = \epsilon i e^{i\theta_1}(u_j^{spac} - u_k^{spac}) + O(\epsilon^2)\,. \tag{11}$$

The controls u_k^{spac}, $1 < k \leq N$, can now be determined to assign formations for the group of particles moving in parallel. As an illustration, we follow the approach proposed in [BL02] to stabilize formations of (non-oriented) particles: the desired formation is specified by the critical points of a scalar potential

$$U = \sum_{j=1}^{N} \sum_{k>i} U_I(r_{jk})$$

where the potential $U_I(r_{jk}) = U_I(r_j - r_k) = U_I(r_{kj})$ determines the desired interaction from particle k on particle j. For instance, the choice

$$U_I(r_{jk}) = \log \|r_{jk}\| + \frac{r_0}{\|r_{jk}\|} \tag{12}$$

results in the feedback

$$\nabla U_I = \frac{\partial U_I}{\partial r_j} = \left(\frac{1}{\|r_{jk}\|} - \frac{r_0}{\|r_{jk}\|^2} \right) \frac{r_{jk}}{\|r_{jk}\|} \tag{13}$$

from particle k on particle j which vanishes only at the equilibrium distance $\|r_{jk}\| = \|r_j - r_k\| = r_0$. Gradient-like dynamics for the slow system (10) are obtained with the feedback control

$$u_k^{spac} = -\sum_{j \neq k} < \nabla U_I(r_{kj}), i e^{i\theta_k} > \tag{14}$$

which causes the scalar potential U to decrease along the solutions in the slow manifold [SPL03]. Equilibrium configurations that minimize the potential U favor uniform spacing between the particles, e.g. see Figure 1.

(a) (b)

Fig. 1. Parallel motion ($K > 0$) for ten vehicles with random initial conditions: (a) without spacing control; (b) with spacing control the group favors formations with uniform spacing between the particles.

5 Circular Motion Design

The design of the particle model on the slow manifold corresponding to $K < 0$, is the design of the original particle model assuming a center of mass at rest [SPL03]. We illustrate how making this assumption simplifies the design of collective motions around a fixed point. The design of the complete particle model is thus decomposed into two steps: (1) the design of a reduced control u^{bal} under the assumption of a fixed center of mass, that is, restricted to the balanced manifold, and (2) the design of a complete control u which reduces to u^{bal} in the balanced manifold and at the same time enforces convergence to the balanced manifold. For $N \geq 3$, the full control is deduced from the reduced control by the expression

$$u = u^{spac} + Ku^{align} = (I - P_\theta(P_\theta^T P_\theta)^{-1} P_\theta^T)u^{bal} + Ku^{align}, \;\; K << 0 \quad (15)$$

where P_θ denotes the $N \times 2$ matrix with first column $\frac{\cos\theta}{N}$ and second column $\frac{\sin\theta}{N}$ [SPL03]. The projector $I - P_\theta(P_\theta^T P_\theta)^{-1} P_\theta^T$ enforces the constraint, $\dot{p}_\theta = 0$, which, in matrix notation, reads $P_\theta^T u^{spac} = 0$.

As an illustration of the above design procedure, we consider the stabilization of the group of particles on a circle of fixed radius, ρ_0, centered at the (fixed) center of mass. The design of a circular motion for a single particle with coordinates (r_k, θ_k) around a fixed beacon R has been addressed in [JK02]. A variant of this feedback control is

$$u_k^{bal} = -f(\rho_k) < \frac{\tilde{r}_k}{\rho_k}, ie^{i\theta_k} > - < \frac{\tilde{r}_k}{\rho_k}, e^{i\theta_k} > \quad (16)$$

with $\tilde{r}_k = r_k - R$ and $\rho_k = \|\tilde{r}_k\|$, as shown in Figure 2.

The second term of the control law (16) stabilizes circular motions: it vanishes when the velocity vector is orthogonal to the relative position vector. The function $f(\cdot)$ in (16) plays the same role as (13) in the parallel control of the previous section, creating an attractive interaction when the distance

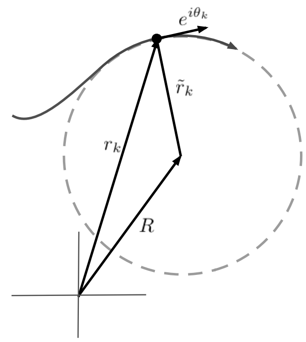

Fig. 2. Circular motion of a single particle around the fixed beacon R

ρ_k exceeds the equilibrium distance ρ_0 and a repulsive interaction otherwise. (The choice $f(\rho_k) = 1 - (\rho_0/\rho_k)^2$ is proposed in [JK03]). With the control (16), the Lyapunov function

$$U_k = -\log | < \frac{\tilde{r}_k}{\rho_k}, ie^{i\theta_k} > | + \int_{\bar{\rho}}^{\rho_k} (f(s) - \frac{1}{s}) ds$$

has a global minimum at a relative equilibrium which corresponds to circular motion around the fixed beacon: the equilibrium is determined by a velocity vector orthogonal to the relative position vector (i.e. $< \tilde{r}_k, e^{i\theta_k} >= 0$), and a distance $\bar{\rho}$ to the beacon such that $f(\bar{\rho}) = \frac{1}{\bar{\rho}}$. Note that u_k^{bal} is nonzero for $\rho_k = \bar{\rho}$ since this is an equilibrium value in the shape coordinates [JK03]. Assuming $\dot{R} = 0$, the time-derivative of U_k satisfies

$$\dot{U}_k = - < \frac{\tilde{r}_k}{\rho_k}, e^{i\theta_k} >^2 / < \frac{\tilde{r}_k}{\rho_k}, ie^{i\theta_k} >$$

The Lyapunov function U_k provides a global convergence analysis for the single particle model. In particular, it can be used to prove asymptotic stability of the clockwise rotation around the fixed beacon.

The design of circular motion for N particles in the balanced manifold is an immediate extension of the single particle design: we apply (16) to each particle $1 \le k \le N$, with the fixed beacon replaced by the center of mass of the group $R = \frac{1}{N} \sum_k r_k$. Under the constraint $\dot{R} = p_\theta = 0$, the Lyapunov function $U = \sum_k U_k$ provides the same global convergence analysis for the group of particles as for a single particle.

Inserting the control law (16) in the general formula (15) thus yields a stabilizing control law for the original particle model (1). The conclusions of the asymptotic analysis, which assumes large values for the parameter $|K|$, seem well retained in simulations even when a time-scale separation is not enforced between the *fast* stabilization of the center of mass and the *slow* stabilization of the circular formation on the balanced manifold, e.g. see Figure 3.

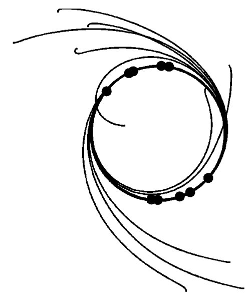

Fig. 3. Circular motion with $K = -1$ and random initial conditions.

6 Conclusion

This paper summarizes a global convergence analysis for the stabilization of relative equilibria by feedback control of self-propelled particles. The proposed approach rests on a two-time scale decomposition of the dynamics that decouples the stabilization of orientation variables from the stabilization of spacing variables. The fast dynamics analysis exploits previous results of the literature for models of phase oscillators. All-to-all sinusoidal coupling of the relative orientations results in stabilization of the linear momentum either to a maximal value (resulting in parallel motion) or to a minimal value (resulting in motion around a fixed center of mass). The slow dynamics analysis also exploits previous results in the literature to achieve parallel motion with prescribed formations or motion of all the particles on a fixed circle.

An important limitation of the present paper is the assumption of all-to-all coupling. This assumption is unrealistic in large groups of organisms and requires a prohibitively demanding communication topology in engineering applications. The results presented are likely to hold under relaxed assumptions on the network connectivity but extending the analysis to these situations is not straightforward. Convergence results for the Viczek model under weak connectivity assumptions [JLM02] provide analysis tools for such generalizations.

The stabilization results can be extended to more challenging modelling and design problems. For example, it can be seen by linearization that certain steering control laws exhibit stability of both parallel and circular group motion for intermediate values of the parameter K, i.e. $|K| < 1$. This bistability is reminiscent of the hysteresis observed in simulations of fish schools, [CKJ$^+$02], albeit via a different mechanism. Furthermore, a library of behavior primitives can be developed based on the parallel and circular motions described here and then used for path planning of vehicle groups. These techniques may prove useful in optimization problems such as area coverage control by a fleet of mobile sensor platforms.

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