Optimal guidance and estimation of a 1D diffusion process by a team of mobile sensors

Sheng Cheng and Derek A. Paley

Abstract— **This paper describes a framework to design guidance for a team of mobile sensors to estimate a distributed parameter system modeled by a diffusion process. The diffusion process has an abstract linear system representation with a linear observation equation, so an infinite-dimensional version of the Kalman filter is applied for estimation. We propose an optimization problem that minimizes the weighted sum of the trace of the covariance operator of the Kalman filter and the guidance effort of the mobile sensors, whose motion is modeled by linear dynamics. This formulation is well-suited for limited endurance mobile sensor platforms. We provide a solution method to solve for the optimal guidance. A finite-dimensional approximation is applied to a simulation in which we analyze how the performance of a single mobile sensor depends on mobility penalty and sensor noise. We also illustrate the application of the framework to a team of heterogeneous sensors.**

I. Introduction

The modern manufacturing industry has benefited from the advantages of (mobile) robots for their reliability, economic efficiency, safety, and ease of use. However, the monitoring and control of large-scale spatiotemporal processes, e.g., oil spills and forest fires, have relied heavily on human operators. These events can pose health threats, cause severe environmental issues, and incur substantial financial costs. Spatiotemporal processes vary in both space and time and, hence, their dynamics can be characterized by partial differential equations (PDEs), e.g., the diffusion equation.

It is generally impossible to measure a system modeled by a PDE, also known as a distributed parameter system (DPS), with a finite number of sensors. Hence, an observer for the DPS is necessary, and various designs have been proposed. Early designs of the observer include least-square methods that can filter and smooth systems governed by linear [1] and nonlinear [2] partial differential equations. For system-theoretical results on the observability of parabolic PDEs, refer to [3], [4].

For sensors placed on the boundary of the spatial domain of a PDE, one may design observers based on boundary measurements. A common approach is backstepping [5], which uses a Volterra transformation to stabilize the observer via a stable target system of the transformation. A Lyapunov-based Luenberger-type observer is proposed in [6] with H^∞ performance constraints for a linear parabolic PDE. Furthermore, optimization techniques, e.g., a linear-quadratic estimator [7], have been proposed for boundary observers using the method of variation.

An infinite-dimensional linear system with additive Gaussian white noise also yields a Kalman filter (KF) that has a similar structure to its finite-dimensional analog. The infinite-dimensional version of the KF first appeared in [8]. Properties of the solution to the operator Riccati differential equation have been discussed in [9], [10]. For numerical approximation and computational issues, [11] provides a summary of approximation results for the infinite-dimensional Riccati equations of a linear-quadratic regulator.

When a network of sensors is deployed for estimating a DPS, a problem arises as to how to place the sensors to yield effective estimation. Such a problem is referred to as sensor placement, for which various optimization criteria have been proposed. The trace of the covariance operator of the KF is a common choice of the objective function to be minimized. [12] proposes a sensor-placement scheme that minimizes this value. A similar problem is investigated in [13], which proves the convergence of the optimal placement computed via finite-dimensional approximation. The same criterion has been applied to sensor placement of the Boussinesq equation [14]. In [15], a randomized observability constant is minimized by choosing suitable shapes and locations of the sensors. Other criteria, e.g., enhanced observability, optimal state estimation, and robust input-output mapping, are discussed for a parabolic PDE in [16].

Geometric approaches can be applied to sensor placement. [17] proposes a scheme that places sensors using the centroidal Voronoi tessellation of the kernel of the observer gain of a parabolic PDE. [18] proposes a method that combines the transfer-function model and geometric rules to design sensor and actuator locations for which high-gain and low-gain proportional feedback control can reduce the influence of pointwise disturbances.

When the sensors are allowed to move, a guidance policy is necessary to take advantage of the additional degree of freedom induced by mobility, which also makes the problem more complicated by introducing the dynamics of the mobile sensors. One may design sensor guidance using the Lyapunov-based method, where the guidance is constructed to make the derivative of the Lyapunov function negative. Representative work is contributed mostly by Demetriou and collaborators [19]–[21]. The Lyapunov-based guidance

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can further be used in a hazardous environment where the regions of high information density reduce sensor life. Such guidance is combined with a switching policy to balance the conflicting needs of information collection and sensor life span [22]. A similar approach uses the gradient of estimation error to guide sensors to the region that has large estimation error [23].

Optimization can also be applied to design sensor guidance. An early work [24] proposes an optimization problem that minimizes the weighted sum of the guidance effort for steering a sensor and mean-square estimation error at a final time. In [25], the sensors are guided to the location that yields a maximum value of the estimation kernel. In [26], receding horizon guidance is proposed to find the sensor path that maximizes mutual information.

This paper proposes an optimization framework that designs guidance for a team of mobile sensors to efficiently estimate a 1D diffusion equation using a centralized Kalman filter. The cost to be minimized is the weighted sum of two terms. One is the trace of the covariance operator, which characterizes the uncertainty of the estimation error, and the other is the guidance effort for steering the mobile sensors. A weight is applied to penalize the guidance effort of the mobile sensors. Our formulation minimizes the Lagrange function—hence, is an intermediate step—of the optimization problem that minimizes the trace of the covariance operator subject to the constraint of upper-bounded guidance effort and sensor dynamics. The problem formulation is particularly motivated by having limited resources (e.g., fuel or batteries) onboard the vehicles that carry the sensors.

Existing results in the literature that are similar to our problem setting include minimizing the trace of the covariance operator plus a cost of guidance [12], [27], [28], the mean-square estimation error at the terminal time plus the weighted guidance effort [24], and the trace of weighted covariance operator [10]. Our approach is different in that the problem in this paper minimizes the trace of the covariance operator plus a quadratic function of the guidance (as the guidance effort).

Our contribution is summarized as follows: this paper (1) formulates an optimization problem that minimizes the trace of the covariance operator combined with the guidance effort for a heterogeneous team of mobile sensors estimating a DPS; (2) shows the conditions of the existence of a solution to the proposed problem; and (3) analyzes via simulation the impact on the performance of the proposed guidance of sensor noise and mobility penalty. The problem studied in this paper is the dual problem of the one studied in [29], which simultaneously designs guidance and actuation of a team of mobile actuators to control a DPS.

The remainder of the paper is organized as follows: Section II introduces the abstract linear-system representation of the diffusion equation, measurement model and dynamics of the sensors, and the infinite-dimensional Kalman filter. Section III states the problem formulation. Section IV introduces the solution method to obtain optimal guidance and its numerical computation using Galerkin approximation. Section V includes the simulation results of parameter studies of a single sensor and a team of heterogeneous sensors. Section VI summarizes the paper and discusses ongoing work.

II. Background

A. Notation and terminology

The paper adopts the following notation. The symbols $\mathbb{R}$ and $\mathbb{R}^+$ denote the set of real numbers and the set of nonnegative real numbers, respectively. The n-nary Cartesian power of a set M is denoted by M^n. An embedding is denoted by $\hookrightarrow$. The space of all bounded linear operators from space X to space Y is denoted by $\mathcal{L}(X,Y)$ or $\mathcal{L}(X)$ if $Y = X$. We define the space of continuous mappings by $C(I,X) = \{F : I \to X \text{ such that } t \mapsto F(t) \text{ is continuous in } \|\cdot\|_X\}$ with the sup norm $\|F(\cdot)\|_{C(I;X)} = \sup_{t\in I}\|F(t)\|_X$. The superscript * denotes an optimal variable, whereas * denotes the adjoint of a linear operator. The transpose of a matrix A is denoted by A^T. An $n \times n$-dimensional diagonal matrix with elements of vector $[a_1, a_2, \ldots, a_n]$ on the main diagonal is denoted by $\text{diag}(a_1, a_2, \ldots, a_n)$. The derivative of a function f evaluated at x is denoted by $\dot{f}(x)$. The trace of an operator $\mathcal{P}$ and a square matrix P is denoted by $\text{Tr}(\mathcal{P})$ and $\text{tr}(P)$, respectively. The ith element of a vector v is $[v]_i$, whereas the element on the ith row and jth column of a matrix V is $[V]_{i,j}$. We follow the terminology of [29]: guidance refers to steering the dynamics of the mobile sensor, whereas control refers to actuation of the dynamics of the DPS.

B. Abstract linear system and associated Kalman filter

Consider the following inhomogeneous diffusion equation over a 1D spatial domain Ω:

$$z_t(x,t) = \alpha z_{xx}(x,t) + D(x,t)w(t), \tag{1}$$

where $x \in \Omega$, $t \in I := [0, t_f]$, and α is the diffusion coefficient, with initial condition $z(x,0) = z_0(x)$ and boundary condition $z(\partial\Omega, t) = 0$. The exact initial condition z_0 is unknown, but its estimate $\hat{z}_0$ is available. The state noise $w(t)$ is a real-valued Gaussian white noise with variance Q and its spatial distribution at time t is specified by $D(\cdot,t) \in L^2(\Omega)$.

Assume linear dynamics for sensor i, whose location ζ_i can be controlled via guidance p_i such that

$$\dot{\zeta}_i(t) = a_i\zeta_i(t) + b_i p_i(t),\ \zeta_i(0) = \zeta_{i0}. \tag{2}$$

We use $a = \text{diag}(a_1, a_2, \ldots, a_m)$, $b = \text{diag}(b_1, b_2, \ldots, b_m)$, $p = (p_1, p_2, \ldots, p_m)^T$, and $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_m)^T$ for conciseness.

The set of guidance U is defined as $U = \{p : p$ is measurable, uniformly bounded by $p_{\text{max}} > 0$, and Lipschitz continuous $|p(t_1) - p(t_2)| \leq c_0|t_1 - t_2|$ for $t_1, t_2 \in I\}$. By the Arzelà–Ascoli theorem, the set U is compact if the distance defined on U is the max-norm such that $d(f,g) = \max_{t\in I}|f(t) - g(t)|$, for $f, g \in U$. This definition is one of several ways to construct a compact set of guidance. We choose this definition because of its clear physical interpretation: the bound p_{max} represents the maximum speed of a vehicle, whereas the Lipschitz coefficient c_0 represents the maximum acceleration of the vehicle.

The measurement can have many types, e.g., pointwise [3], [4], [24], interval integral [10], [21], interval average [30], and Gaussian-type kernel [10]. Assume each sensor measures an interval average of the state. Define the indicator function $\mathbb{1}$ as

$$\mathbb{1}_{[a,b]}(x) = \begin{cases} 1, \text{ if } a \le x \le b \\ 0, \text{ otherwise} \end{cases}. \tag{3}$$

Use $\mathbb{B}_{x_0,r}(\cdot) \in L^2(\Omega)$ to denote the interval average centered at x_0 with radius r such that

$$\mathbb{B}_{x_0,r}(x) = \frac{1}{2r}\mathbb{1}_{[x_0-r,x_0+r]}(x). \tag{4}$$

The measurement $y(\cdot) \in \mathbb{R}^m$ is

$$y(t) = \int_\Omega \mathbb{B}_{\zeta(t),r}(x)z(x,t)\mathrm{d}x + v(t), \tag{5}$$

where $\mathbb{B}_{\zeta(t),r}$ is a vectorized representation such that

$$\mathbb{B}_{\zeta(t),r} := [\mathbb{B}_{\zeta_1(t),r}, \mathbb{B}_{\zeta_2(t),r}, \dots, \mathbb{B}_{\zeta_m(t),r}]^T. \tag{6}$$

The measurement noise $v(t)$ is a vector of Gaussian white noise with covariance $R := \mathrm{diag}([\sigma_1^2, \sigma_2^2, \dots, \sigma_m^2])$ and σ_i^2 is the variance of the Gaussian white noise of sensor $i \in \{1, 2, \dots, m\}$.

For simplicity, represent PDE state in (1) by an abstract linear system whose state variable $\mathcal{Z}(t)$ represents $z(\cdot,t)$ such that

$$\begin{cases} \dot{\mathcal{Z}}(t) = \mathcal{A}\mathcal{Z}(t) + \mathcal{D}(t)w(t) \\ y(t) = \mathcal{C}^\star_{\zeta(t)}\mathcal{Z}(t) + v(t) \end{cases}, \tag{7}$$

where $\mathcal{Z}$ belongs to a Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Here, the variable $\mathcal{Z}(t)$ is the state of the DPS and space $\mathcal{H} = L^2(\Omega)$ is the state space. The operator $\mathcal{A}$ is defined as $\mathcal{A}\psi = \alpha\partial^2\psi(x)/\partial x^2$ with $\psi \in \mathrm{Dom}(\mathcal{A}) = \{\psi \in H_0^1(\Omega),\ \nabla^2\psi \in L^2(\Omega)\} = H^2(\Omega) \cap H_0^1(\Omega)$ [31]. The output operator $\mathcal{C}^\star_{\zeta(t)} \in \mathcal{L}(\mathcal{H}, \mathbb{R}^m)$ specifies measurement of the state such that $\mathcal{C}^\star_{\zeta(t)}\psi = \int_\Omega \mathbb{B}_{\zeta(t),r}(x)\psi(x)\mathrm{d}x$ for all $\psi \in \mathcal{H}$. The operator $\mathcal{D}(\cdot) \in L^2(I, \mathcal{L}(\mathbb{R}, \mathcal{H}))$ is the operator version of $D(\cdot,\cdot)$ in (1).

Definition 1 (Definition 4.5 of [10]): Let I be a real interval and $\mathcal{F}(\cdot) : \Omega^m \to \mathcal{L}(L^2(\Omega), \mathbb{R}^m)$ be of the form $[\mathcal{F}(\bar{x})\phi]_i = \int_\Omega K(x, [\bar{x}]_i)\phi(x)\mathrm{d}x$, where $K(\cdot, [\bar{x}]_i) \in L^2(\Omega)$ is the integral kernel. We say that $\mathcal{F}(\bar{x})$ is continuous with respect to location if there is a continuous function $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that $g(0) = 0$ and $\|K(\cdot,x) - K(\cdot,y)\|_{L^2(\Omega)} \le g(\|x-y\|_\mathbb{R}),\ \forall x, y \in \Omega$.

Remark 1: The interval average operator $\mathcal{C}^\star_{\zeta(t)}$ is continuous with respect to location, where $g(u) = (u/2r^2)^{1/2}$ in Definition 1.

Analogous to a finite-dimensional linear system, the infinite-dimensional linear system (7) admits a Kalman filter (KF). For the derivation of the KF of an abstract linear system, one may refer to [1], [32]. The linear quadratic optimal estimation $\hat{\mathcal{Z}}(t)$ of the state $\mathcal{Z}(t)$ can be updated from the measurement $y(t)$ by

$$\dot{\hat{\mathcal{Z}}}(t) = \mathcal{A}\hat{\mathcal{Z}}(t) + \mathcal{P}(t)\mathcal{C}_{\zeta(t)}R^{-1}(y(t) - \hat{y}(t)), \tag{8}$$
$$\hat{\mathcal{Z}}(t_0) = \hat{\mathcal{Z}}_0, \tag{9}$$

where $\hat{y}(t) = \mathcal{C}^\star_{\zeta(t)}\hat{\mathcal{Z}}(t)$ is the observation of the estimated system. Let $\bar{\mathcal{C}}_\zeta\bar{\mathcal{C}}^\star_\zeta(t)$ be the compact representation for $\mathcal{C}_{\zeta(t)}R^{-1}\mathcal{C}^\star_{\zeta(t)}$. The covariance $\mathcal{P}$ is propagated forward in time by solving the following operator Riccati equation:

$$\begin{aligned}\dot{\mathcal{P}}(t) = {} & \mathcal{A}\mathcal{P}(t) + \mathcal{P}(t)\mathcal{A}^\star + \mathcal{D}(t)Q\mathcal{D}^\star(t) \\ & - \mathcal{P}(t)\bar{\mathcal{C}}_\zeta\bar{\mathcal{C}}^\star_\zeta(t)\mathcal{P}(t),\end{aligned} \tag{10}$$

Assume initial condition $\mathcal{P}(0)$ is given as the covariance operator $\mathcal{P}_0$ of the initial estimation error $\mathcal{Z}(0) - \hat{\mathcal{Z}}(0)$.

The covariance operator $\mathcal{P}$ characterizes the uncertainty of the estimation error. Specifically, consider the trace operator $\mathrm{Tr}(\cdot) : \mathcal{L}(\mathcal{H}) \mapsto \mathbb{R}$ defined as follows:

$$\mathrm{Tr}(\Pi) = \sum_{i=1}^\infty \langle\phi_i, \Pi\phi_i\rangle, \qquad \Pi \in \mathcal{L}(\mathcal{H}), \tag{11}$$

where $\{\phi_i\}_{i=1}^\infty$ is an arbitrary orthonormal basis that spans $\mathcal{H}$. Notice that $\mathrm{Tr}(\cdot)$ is independent of the choice of the orthonormal basis [10]. The expected value of the squared norm of the estimation error is the trace of the covariance operator $\mathcal{P}$ [10], [13]:

$$\mathrm{Tr}(\mathcal{P}(t)) = \mathbb{E}\big[\|\mathcal{Z}(t) - \hat{\mathcal{Z}}(t)\|^2\big]. \tag{12}$$

Definition 2 (Definition 3.2 of [10]): Let $\mathbb{H}$ be a separable complex Hilbert space. For $1 \le p < \infty$, let $\mathcal{J}_p(\mathbb{H})$ denote the set of all bounded operators $\mathcal{L}(\mathbb{H})$ such that $\mathrm{Tr}(|A|^p) < \infty$, where $|A| := \sqrt{A^\star A}$. If $A \in \mathcal{J}_p(\mathbb{H})$, then the $\mathcal{J}_p$-norm of A is defined as $\|A\|_p := (\mathrm{Tr}(|A|^p))^{1/p} < \infty$.

The class $\mathcal{J}_1(\mathbb{H})$ and $\mathcal{J}_2(\mathbb{H})$ are known as the space of trace operators and the space of Hilbert-Schmidt operators, respectively. Note that a continuous embedding $\mathcal{J}_{p_1}(\mathbb{H}) \hookrightarrow \mathcal{J}_{p_2}(\mathbb{H})$ holds if $1 \le p_1 < p_2 \le \infty$. In other words, if $A \in \mathcal{J}_{p_1}(\mathbb{H})$, then $A \in \mathcal{J}_{p_2}(\mathbb{H})$ and $\|A\|_{p_2} \le \|A\|_{p_1}$.

Assume the initial condition $\mathcal{P}_0 \in \mathcal{J}_1(\mathcal{H})$ and $\mathcal{P}_0 \ge 0$. By Lemma 4.3 of [10], $\mathcal{D}(\cdot)Q\mathcal{D}^\star(\cdot) \in L^1(I, \mathcal{J}_1(\mathcal{H}))$. Since $\mathbb{B}_{\zeta(t),r} \in L^2(\Omega)$ and $\mathcal{C}^\star_{\zeta(t)}$ is continuous with respect to location, by Lemma 4.6 of [10], $\bar{\mathcal{C}}_\zeta\bar{\mathcal{C}}^\star_\zeta(\cdot) \in C(I, \mathcal{J}_1(\mathcal{H}))$, which implies $\bar{\mathcal{C}}_\zeta\bar{\mathcal{C}}^\star_\zeta(\cdot) \in C(I, \mathcal{L}(\mathcal{H}))$. By Theorem 5.1 of [10], the Riccati equation (10) yields a unique weak solution in $C(I, \mathcal{J}_1(\mathcal{H}))$. Moreover, by Theorem 6.2 of [10], there exists a finite-dimensional approximation $\mathcal{P}_N \in C(I, \mathcal{J}_1(\mathcal{H}))$ of $\mathcal{P}$ such that

$$\sup_{t\in I} \|\mathcal{P}(t) - \mathcal{P}_N(t)\|_1 \to 0 \tag{13}$$

as $N \to \infty$.

III. Problem formulation

The goal is to find guidance $p(\cdot)$ under which multiple heterogeneous sensors can reduce the uncertainty of the estimate of a DPS. The uncertainty is characterized by the trace of the covariance operator, integrated over the horizon $I := [0, t_f]$.

Meanwhile, the guidance effort is constrained indirectly using a penalty term in the cost function. The optimization problem is formulated as follows:

$$\begin{aligned} &\underset{p(t)\in U}{\text{minimize}} && \int_0^{t_f} \mathrm{Tr}(\mathcal{P}(t)) + \frac{1}{2}p^T(t)\gamma p(t)\mathrm{d}t \\ &\text{subject to} && \dot{\zeta}(t) = a\zeta(t) + bp(t),\ \zeta(0)=\zeta_0, \end{aligned} \tag{P}$$

where $\gamma \in \mathbb{R}^{m\times m}$ is positive definite and can be chosen to address resource constraints, such as limited fuel or battery life. Since the set U is compact, a solution exists for problem (P) when the cost function is continuous with respect to p. (The proof is omitted for space constraints.)

Problem (P) can be applied to the case of limited onboard resources of each mobile sensor when γ is diagonal. This problem minimizes the Lagrangian function of (and, hence, is an intermediate step to solve) the optimization problem that minimizes the integral of the trace of the covariance operator $\mathcal{P}$ subject to the constraints of bounded guidance effort and linear dynamics of the mobile sensors.

IV. Multisensor optimal guidance

A method to solve problem (P) uses Pontryagin's maximum principle. Consider the Hamiltonian

$$H(t) = \mathrm{Tr}(\mathcal{P}(t)) + \frac{1}{2}p^T(t)\gamma p(t) + \lambda^T(t)(a\zeta(t) + bp(t)),$$

where $\lambda(t) \in \mathbb{R}^m$ is the costate associated with $\zeta(t)$. The necessary conditions of (local) optimality are as follows:

$$\dot{\zeta}^*(t) = a\zeta^*(t) + bp^*(t),\ \zeta^*(0)=\zeta_0, \tag{14a}$$
$$\dot{\lambda}^*(t) = -a^T\lambda^*(t) - (\nabla_{\zeta^*}\mathrm{Tr}(\mathcal{P}(t)))^T,\ \lambda^*(t_f)=0, \tag{14b}$$
$$p^*(t) = -\gamma^{-1}b^T\lambda^*, \tag{14c}$$

where $\nabla_\zeta \mathrm{Tr}(\mathcal{P}(t))$ is the gradient of $\mathrm{Tr}(\mathcal{P}(t))$ with respect to sensors' locations $(\zeta_1(t), \zeta_2(t), \dots, \zeta_m(t))$. We have

$$[\nabla_\zeta \mathrm{Tr}(\mathcal{P}(t))]_i = \frac{\partial \mathrm{Tr}(\mathcal{P}(t))}{\partial \zeta_i(t)}. \tag{15}$$

The necessary condition (14) essentially requires the solution to a two-point boundary value problem, which further requires the derivation of $\nabla_\zeta \mathrm{Tr}(\mathcal{P}(t))$.

By [10, Theorem 5.5], the Fréchet derivative $\Lambda(t)$ of $\mathcal{P}(t)$ with respect to $\bar{\mathcal{C}}_\zeta \bar{\mathcal{C}}_\zeta^\star(t)$ is the unique solution to

$$\begin{aligned} \Lambda h(t) = & -\int_0^t S(t-s)\big((\Lambda h)\bar{\mathcal{C}}_\zeta\bar{\mathcal{C}}_\zeta^\star\mathcal{P} + \mathcal{P}\bar{\mathcal{C}}_\zeta\bar{\mathcal{C}}_\zeta^\star(\Lambda h) \\ & + \mathcal{P}h\mathcal{P}\big)(s)S^\star(t-s)\mathrm{d}s, \end{aligned} \tag{16}$$
$$\Lambda(0) = 0, \tag{17}$$

for all $h \in C(I, \mathcal{J}_1(\mathcal{H}))$ and all $t \in I$, where $S(t)$ is the C_0-semigroup generated by $\mathcal{A}$. We use the chain rule and the Fréchet derivative $\Lambda(t)$ to derive the gradient $\nabla_\zeta \mathrm{Tr}(\mathcal{P}(t))$:

$$[\nabla_\zeta \mathrm{Tr}(\mathcal{P}(t))]_i = \mathrm{Tr}(\Lambda(t) \circ D_{\zeta_i(t)}\bar{\mathcal{C}}_\zeta\bar{\mathcal{C}}_\zeta^\star(t)), \tag{18}$$

where $D_{\zeta_i(t)}\bar{\mathcal{C}}_\zeta\bar{\mathcal{C}}_\zeta^\star(t)$ is the Frèchet derivative of the operator $\bar{\mathcal{C}}_\zeta\bar{\mathcal{C}}_\zeta^\star(t)$ with respect to location $\zeta_i(t)$ of sensor i.

A finite-dimensional approximation of the infinite-dimensional state estimation $\hat{\mathcal{Z}}(t)$ and covariance operator $\mathcal{P}(t)$ is necessary for numerical computation. Consider a finite-dimensional subspace $\mathcal{H}_N \subset \mathcal{H}$ with dimension N. The inner product and norm of $\mathcal{H}_N$ are inherited from that of $\mathcal{H}$. Let $T_N : \mathcal{H} \to \mathcal{H}_N$ denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_N$. We adopt the Galerkin approximation scheme which satisfies the standard assumption [33], [34] such that for all ϕ in the Sobolev space $H_0^1(\Omega)$,

$$\lim_{N\to\infty} \|T_N\phi - \phi\|_{H_0^1(\Omega)} = 0. \tag{19}$$

Let $\{\phi_k(\cdot)\}_{k=1}^N$ be a collection of orthonormal basis functions in $\mathcal{H}$. Let $\Phi_N(\cdot)$ be a column vector of this base such that $[\Phi_N(\cdot)]_k = \phi_k(\cdot)$ for $k = 1, 2, \dots, N$. Let $\hat{\mathcal{Z}}(t) = \Phi_N^T(\cdot)\hat{z}_N(t)$, where the equality holds in the weak sense such that

$$\int_\Omega \hat{\mathcal{Z}}(t)\psi(x)\mathrm{d}x = \int_\Omega \Phi_N^T(x)\hat{z}_N(t)\psi(x)\mathrm{d}x, \qquad \forall\psi \in \mathcal{H}.$$

The operator $\mathcal{P}(t) \in \mathcal{J}_1(\mathcal{H})$ is also a Hilbert-Schmidt operator in $\mathcal{J}_2(\mathcal{H})$, which admits a kernel representation $p(x, y, t)$ such that [35]

$$(\mathcal{P}(t)\psi)(x) = \int_\Omega p(x,y,t)\psi(y)\mathrm{d}y. \tag{20}$$

Let $P_N(t)$ denote the Galerkin approximation of the kernel $p(\cdot,\cdot,t)$ such that $p(x,y,t) = \Phi_N^T(x)P_N(t)\Phi_N^T(y)$ holds in the weak sense.

The observation $\hat{y}$ of the estimated state at the sensor location is computed by

$$\begin{aligned} \hat{y}(t) = \mathcal{C}_\zeta^\star\hat{\mathcal{Z}}(t) &\approx \int_\Omega \mathbb{B}_{\zeta(t),r}(x)\Phi_N^T(x)\hat{z}_N(t)\mathrm{d}x \\ &= \frac{1}{2r}\big(\mathring{\Phi}_N^T(\zeta(t)+r\mathbf{1}) - \mathring{\Phi}_N^T(\zeta(t)-r\mathbf{1})\big)\hat{z}_N(t), \end{aligned} \tag{21}$$

where $\mathring{\Phi}_N$ is the primitive function of Φ_N such that $\mathring{\Phi}_N(x) = \int_0^x \Phi_N(y)\mathrm{d}y$ for a scalar-valued x and $\mathbf{1}$ is an m-dimensional column vector with all elements being 1. Here, Φ_N (and also $\mathring{\Phi}_N$) admits a vectorized representation such that for $i \in \{1, 2, \dots, N\}$ and $j \in \{1, 2, \dots, m\}$

$$[\Phi_N(\zeta(t)^T)]_{i,j} = [\Phi_N^T(\zeta(t))]_{j,i} = \phi_i(\zeta_j(t)).$$

The Galerkin approximation of (8) is

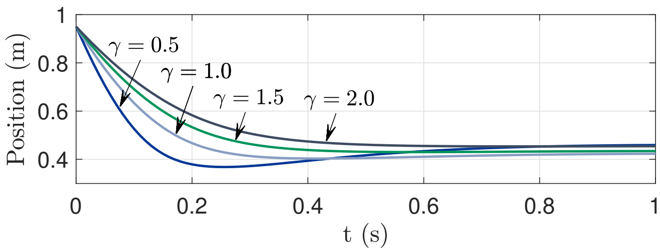
$$\begin{aligned} \dot{\hat{z}}_N(t) = & -L_N^{-1}M_N\hat{z}_N(t) + \frac{1}{2r}P_N\Big[\mathring{\Phi}_N(\zeta(t)^T + r\mathbf{1}^T) \\ & - \mathring{\Phi}_N(\zeta(t)^T - r\mathbf{1}^T)\Big]R^{-1}(y(t) - \hat{y}(t)), \end{aligned} \tag{22}$$

where $M_N \in \mathbb{R}^{N\times N}$ and $L_N \in \mathbb{R}^{N\times N}$ are such that

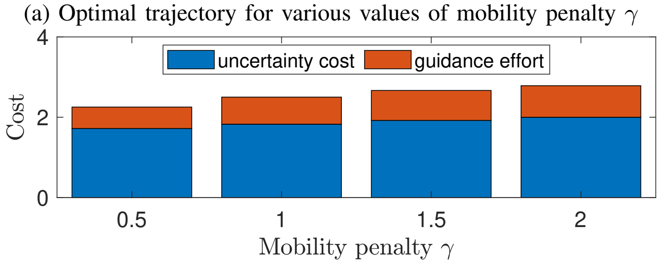
$$M_N = \int_\Omega \dot{\Phi}_N(x)\dot{\Phi}_N^T(x)\mathrm{d}x,\ L_N = \int_\Omega \Phi_N(x)\Phi_N^T(x)\mathrm{d}x.$$

The numerical computation of $\mathrm{Tr}(\mathcal{P}(t))$ is

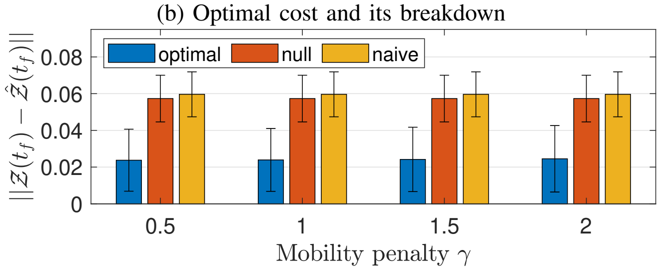
$$\begin{aligned} \mathrm{Tr}(\mathcal{P}(t)) =& \sum_{k=1}^\infty \langle\phi_k, \mathcal{P}(t)\phi_k\rangle \\ \approx& \mathrm{tr}\big((\sum_{k=1}^N e_k e_k^T)P_N(t)\big) = \mathrm{tr}(P_N(t)), \end{aligned} \tag{23}$$



(a) Optimal trajectory for various values of mobility penalty γ

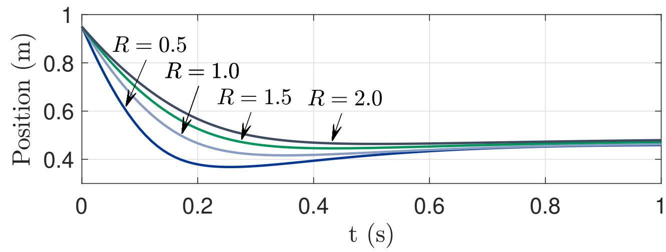


(b) Optimal cost and its breakdown

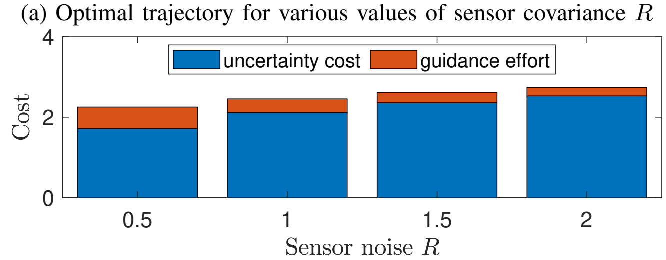


(c) Norm of the terminal estimation error. The color bar shows the mean value; the error bar shows the standard deviation.

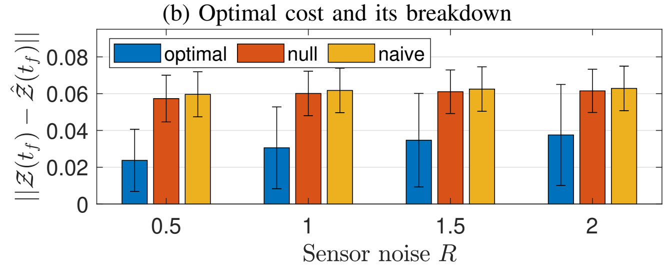
Fig. 1: The value of sensor noise R is fixed at 0.5 while mobility penalty γ takes values in the range $\{0.5, 1, 1.5, 2\}$.



(a) Optimal trajectory for various values of sensor covariance R



(b) Optimal cost and its breakdown



(c) Norm of the terminal estimation error. The color bar shows the mean value; the error bar shows the standard deviation.

Fig. 2: The value of mobility penalty γ is fixed at 0.5 while sensor noise R takes values in the range of $\{0.5, 1, 1.5, 2\}$.

where e_k is an N-dimensional zero vector except 1 on its kth row.

Denote the finite-dimensional approximation of $\Lambda(t)$ by $H_N(t)$, then

$$\mathrm{Tr}(\Lambda(t) \circ D_{\zeta_i(t)} \bar{\mathcal{C}}_\zeta \bar{\mathcal{C}}_\zeta^\star(t)) \approx -\frac{1}{4r^2} \mathrm{tr}(H_N(t)(\Gamma + \Gamma^T)),$$

where

$$\Gamma = \big(\Phi_N(\bar{\zeta}_i(t)^T + r\mathbf{1}^T) - \Phi_N(\bar{\zeta}_i(t)^T - r\mathbf{1}^T)\big) R^{-1} \big(\mathring{\Phi}_N^T(\zeta(t) + r\mathbf{1}) - \mathring{\Phi}_N^T(\zeta(t) - r\mathbf{1})\big) \tag{24}$$

and we use the bar notation $\bar{\zeta}_i$ together with the subscript i to denote an m-dimensional zero vector except for the ith row being ζ_i.

V. Simulation Results

This section shows the simulation results for a single sensor and a team of heterogeneous sensors. Comparison and analysis are made regarding the estimation performance of the mobile sensor(s) under the optimal guidance. We use the sinusoidal basis $\{\phi_n(x) = \sqrt{2}\sin(\pi n x),\ x \in \Omega = [0, 1],\ n \geq 1\}$ as the orthonormal basis for $\mathcal{H}$. The parameters in the simulation are as follows:

$$z_0(x) = 4(x - x^2),\ \hat{z}_0(x) = \mathrm{erf}(50x)\mathrm{erf}(50 - 50x),$$
$$D(x) = \sin(\pi x)e^{-2x},\ p(x, y, 0) = 5\chi(x, y), Q = 0.01$$
$$t_f = 1, \alpha = 0.1, N = 24, r = 0.05, a_i = 0, b_i = 1,\ \text{for all } i,$$

where $\chi(x, y) = 1$ if $x = y$ and $\chi(x, y) = 0$ if $x \neq y$. The initial guess $\hat{z}_0$ is chosen as the smoothed rectangular function with unit value on Ω. And the kernel of the covariance operator $p(x, y, 0)$ is chosen to be arbitrarily large.

We use the shooting method [36] to find a (local) optimal solution satisfying (14). The MATLAB function `fsolve` is applied to search for the initial value $\lambda(0)$ such that (14) is met. The forward propagation of (14a) and (14b) is done via the Runge-Kutta method.

A. Single sensor results

Two important parameters in the problem setting are the sensor noise covariance R and mobility penalty γ. Smaller R yields higher sensor quality, whereas smaller γ yields better mobility of the vehicle. For example, if γ is the mass of the vehicle, then the guidance effort is the kinetic energy of the vehicle. These parameters influence the performance of the estimation, as shown next. The terminologies *uncertainty cost* and *guidance effort* refer to $\int_0^{t_f} \mathrm{Tr}(\mathcal{P}(t))\mathrm{d}t$ and $\frac{1}{2}\int_0^{t_f} p(t)^T \gamma p(t)\mathrm{d}t$, respectively.

First, hold either R or γ fixed and vary the other to see the variation of the optimal trajectory. Fig. 1a displays the trajectories when $R = 0.5$ and γ varies from 0.5 to 2. A clear tendency of less maneuvering of the trajectory can be observed as γ increases. Fig. 1b displays the amount of uncertainty cost and guidance effort, which both grow as γ increases, yielding performance reduction evaluated under the metric of problem (P).

Monte Carlo simulations were conducted on four test cases. Moreover, we compare the optimal guidance with naive guidance (under which the sensor traverses the spatial domain with a trigonometric trajectory) and null guidance (under which the sensor is stationary). The mean and standard deviation of the norm of the terminal estimation error are shown in Fig. 1c. Observe that estimation error grows when γ increases. Compared with the naive guidance and the null guidance, the sensor under optimal guidance significantly reduces the estimation uncertainty.

Fig. 2a displays the trajectories when $\gamma = 0.5$ and R varies from 0.5 to 2. A tendency of less maneuvering also appears as R increases, similar to the case of increasing γ with fixed R in Fig. 1a. The uncertainty cost and total cost grow notably, as displayed in Fig. 2b, when R increases. The results of Monte Carlo simulation for these four cases are shown in Fig. 2c. The optimal guidance performs better than the naive guidance and the null guidance when evaluated by the mean value. However, both the mean and standard deviation of the optimal guidance grow as R increases. This tendency does not appear in Fig. 1c, where R is fixed while γ increases. The explanation is that the norm of the terminal estimation error is affected directly by the sensor noise level (through (8) and (10)) and indirectly by mobility penalty (through the dynamics (14) and (10)). Therefore, increasing R and holding γ fixed yields apparent growth of the terminal estimation error compared with increasing γ and holding R fixed.

B. Team of heterogeneous sensors

The parameters R and γ essentially relate to operational planning: one may invest more for better sensor quality or a swifter vehicle. Similarly, one may invest more for a team of superior mobile sensors ($R = 0.5$ and $\gamma = 0.5$) than a team of poor mobile sensors ($R = 2$ and $\gamma = 1$). The latter has twice as much sensor noise (in terms of standard deviation) and twice mobility penalty as the former. One may balance the conflicting needs of performance and investment by deploying a team of heterogeneous sensors, i.e., a mixed team of superior and poor sensors. The following simulation compares the performance of a team of heterogeneous sensors (m_p poor sensors and $8-m_p$ superior sensors, for m_p in the range of $\{1, 2, \dots, 7\}$) with that of a homogeneous team (eight superior sensors). To adapt to a total of eight sensors, we adjust Q to 0.64, of which the standard deviation is eight times the one in the case of a single sensor. The sensors are evenly distributed in the interval $[0.9, 0.95]$ at $t = 0$. Fig. 3 shows the normalized uncertainty cost $(J^u_{m_p} - J^u_0)/J^u_0$, where $J^u_{m_p}$ and J^u_0 denote

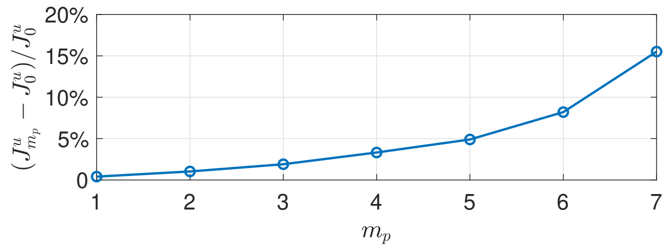


Fig. 3: Normalized uncertainty cost of the heterogeneous team with $8 - m_p$ superior mobile sensors and m_p poor mobile sensors.

the uncertainty cost of the heterogeneous team and that of the homogeneous team, respectively. The performance of the heterogeneous team, evaluated by the normalized uncertainty cost, is inferior to that of the homogeneous team: performance degrades as m_p increases. However, the degradation is kept within 20% even when seven superior sensors are replaced by poor sensors. Meanwhile, the investment reduces linearly as m_p increases, indicating the cost effectiveness of the heterogeneous team.

VI. Conclusion

This paper proposes a guidance design method for a team of mobile sensors to estimate a linear distributed parameter system modeled by a diffusion equation. We formulate an optimization problem that minimizes the weighted sum of the trace of the covariance operator of the Kalman filter and the guidance effort of the mobile sensors. We use Pontryagin's maximum principle together with Galerkin approximation to numerically compute optimal guidance. Simulation results show that, for a single sensor, either improving sensor quality or reducing mobility penalty improves the performance evaluated by the proposed cost function and terminal estimation error. For a team of mobile sensors, degradation of the estimation performance is kept within 20% when several superior sensors are replaced by the poor alternatives, indicating the cost effectiveness of a heterogeneous team.

Ongoing and future work includes extending the framework to a diffusion process in a 2D domain; proving the conditions under which the guidance computed from the proposed numerical method converges to the optimal guidance of the proposed problem; and accelerating the numerical computation using emerging technologies, e.g., via a high-performance graphics processing unit.

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