Collective Motion of Self-Propelled Particles: Stabilizing Symmetric Formations on Closed Curves

Derek A. Paley[1] and Naomi Ehrich Leonard[2]
Mechanical and Aerospace Engineering
Princeton University
Princeton, NJ 08544, USA
{dpaley}{naomi}@princeton.edu

Rodolphe J. Sepulchre[3]
Electrical Engineering and Computer Science
Université de Liège
Institut Montefiore B28, B-4000 Liège, Belgium
r.sepulchre@ulg.ac.be

***Abstract*— We provide feedback control laws to stabilize formations of multiple, unit speed particles on smooth, convex, and closed curves with definite curvature. As in previous work we exploit an analogy with coupled phase oscillators to provide controls which isolate symmetric particle formations that are invariant to rigid translation of all the particles. In this work, we do not require all particles to be able to communicate; rather we assume that inter-particle communication is limited and can be modeled by a fixed, connected, and undirected graph. Because of their unique spectral properties, the Laplacian matrices of circulant graphs play a key role. The methodology is demonstrated using a superellipse, which is a type of curve that includes circles, ellipses, and rounded rectangles. These results can be used in applications involving multiple autonomous vehicles that travel at constant speed around fixed beacons.**

I. INTRODUCTION

Cooperative control of groups has a number of applications including the design of mobile sensor networks for ocean monitoring [1]. Motivated by this work, we derive feedback control laws to stabilize collective motion of a group of individuals in symmetric formations. We model the particles as point masses moving at constant speed in the plane subject to identical gyroscopic controls, after [2]. In this setting, the configuration of each particle is described by its position and direction. As in earlier work, [3], we exploit an analogy between the particle model and a system of coupled phase oscillators. We extend this work by providing controls to isolate symmetric formations of particles on smooth, closed, and convex curves with definite curvature.

In a phase oscillator model, each phase evolves according to its natural frequency. A constant natural frequency in the phase model corresponds to a constant turning rate in the particle model, which drives the particle around a circle of fixed radius. In this paper, we derive the non-constant natural frequency in the phase model that corresponds to the desired closed curve in the particle model. Indeed, the turning rate which drives a particle at unit speed along a specific curve is the local curvature of the desired trajectory.

[1]This work is supported by the National Science Foundation Graduate Research Fellowship, the Princeton University Gordon Wu Graduate Fellowship, and the Pew Charitable Trust grant 2000-002558.

[2]Research partially supported by ONR grants N00014–02–1–0826, N00014–02–1–0861, and N00014–04–1–0534.

[3]This paper presents research results of the Belgian Programme on Interuniversity Attraction Poles, initiated by the Belgian Federal Science Policy Office. The scientific responsibility rests with its authors.

Symmetric formations of the particles are configurations in which the particles are symmetrically distributed around the same curve as they move. As in our previous work, we extend our main result by identifying and breaking various symmetries. In the closed-loop particle model, if the controls depend only on relative position and direction, then the system is invariant to rigid translation and rotation of all the particles [2], [4]. We break the rotation symmetry by choosing the orientation of the particle (non-circular) orbits. We break the translation symmetry by introducing reference beacons about which to stabilize the particle orbits. Lastly, we break the particle permutation symmetry by limiting each particle's control law to be a function of the relative positions and directions of only a few other particles.

The recent control literature on collective motion is quite extensive. In particular, we have found [5] and [6] to be closely related to our work on coupled phase models described by interconnection graphs. The class of circulant graphs is significant to our work due to well known properties of its spectrum, see e.g. [7] and [8]. In other related work, [9], [10], coordinated controls for a multiple particle system are developed to track smooth, closed curves of arbitrary shape using curvature and arc length as feedback.

The outline of the paper is as follows. In Section II, we introduce the particle model and the relevant graph theory. In Section III, we define the curve-phase model, which describes the progress of each particle around a curve by a phase angle. In Section IV, we provide a translation invariant control law to steer multiple particles around the same curve. In Section V, we introduce the design methodology for controlling curve-phase which is used in Section VI to isolate symmetric patterns of the particle curve-phases. In Section VII, we provide a control law to stabilize symmetric formations of particles on the same curve. In Section VIII, we describe extensions to the main result and briefly describing ongoing work.

II. PARTICLE MODEL

Consider the following second-order, planar *particle model* in which each particle moves at a constant (unit) speed [2],

$$\begin{aligned}\dot{r}_k &= e^{i\theta_k}\\ \dot{\theta}_k &= u_k,\ k = 1, \ldots, N,\end{aligned} \tag{1}$$

where $r_k = x_k + iy_k \in \mathbb{C} \equiv \mathbb{R}^2$ and $\theta_k \in \mathbb{T} \equiv S^1$ are the kth particle position and direction, respectively, and u_k is the steering control[1]. We often refer to the direction of motion of each particle θ_k as its phase. An equivalent description of the model (1) is a system of N first-order, planar rigid bodies in which each body moves at constant speed subject to the non-holonomic constraint that its velocity is strictly aligned with its heading θ_k.

In this paper we do not assume that all particles can communicate with (or sense) one another. We describe the particle limited communication topology by a fixed, connected, and undirected graph. Each node (vertex) in the graph corresponds to a particle and each edge corresponds to a communication link between two particles. Let d_k be the degree of the kth node in the interconnection graph and let $\mathcal{N}_k$ denote the set of vertices (neighbors) connected to vertex k. The Laplacian matrix of the interconnection graph is $L = D - A$, where the degree and adjacency matrices are given by $D = \mathrm{diag}(\mathbf{d})$ and $[A]_{k,j} = 1$ if $j \in \mathcal{N}_k$ and zero otherwise. Under these assumptions, the Laplacian has the following properties that we use below [11]:

P1) $L = L^T$ is symmetric and all its eigenvalues are real and non-negative;

P2) $\mathrm{rank}(L) = N - 1$ and the eigenvector corresponding to zero is $\mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^N$;

P3) $L = BB^T$, where $B \in \mathbb{R}^{N \times e}$ is the incidence matrix of the graph and e is the number of edges in the graph.

Let L be the Laplacian of a connected graph. Using Properties P1 and P2, the quadratic form $Q(\mathbf{z}) = \frac{1}{2} < \mathbf{z}, L\mathbf{z} >$ vanishes only when $\boldsymbol{z} = \mathbf{1}z_0$ [6]. $Q(\mathbf{z})$ defines a norm on the shape space $\mathbb{C}^N/\mathbb{C}$ induced by the action of the group of rigid displacements $\boldsymbol{z} \mapsto \boldsymbol{z} + \mathbf{1}z_0$ [12].

III. CURVE-PHASE MODEL

Let $\phi : \mathbb{T} \to [0, 2\pi)$, $\theta \mapsto \phi(\theta)$, be a smooth map and $\rho : [0, 2\pi) \to \mathbb{C}$, $\phi \mapsto \rho(\phi)$, be a parametrization of a smooth, closed convex curve, C, with definite curvature. The tangent vector to C is $\frac{d\rho}{d\phi} \in \mathbb{C}$. The velocity constraint $\frac{d\rho}{d\phi} = \left|\frac{d\rho}{d\phi}\right| e^{i\theta}$ is satisfied if θ is the angle of the vector tangent to $\rho(\phi)$. The arc length along C, $\sigma : [0, 2\pi) \to \mathbb{R}^+$, is defined by

$$\sigma(\phi) = \int_0^{\phi} \left|\frac{d\rho}{d\bar{\phi}}\right| d\bar{\phi}. \quad (2)$$

For a curve C that satisfies the velocity constraint, the local curvature of C, $\kappa : [0, 2\pi) \to \mathbb{R}$, is defined by

$$\kappa(\phi) = \pm\frac{d\theta}{d\sigma}, \quad (3)$$

where the sign determines the sense of rotation. By assumption, the curvature of C is bounded and definite, i.e. $0 < |\kappa(\phi)| < \infty$. Using (2) and (3), we obtain

$$\kappa^{-1}(\phi) = \frac{1}{\kappa(\phi)} = \pm\frac{d\sigma}{d\theta} = \pm\frac{d\sigma}{d\phi}\frac{d\phi}{d\theta} = \pm\left|\frac{d\rho}{d\phi}\right|\frac{d\phi}{d\theta}. \quad (4)$$

[1]We will use the following notational conventions. We drop the subscript and use bold to represent a vector of length N such as $\mathbf{r} = (r_1, \ldots, r_N)^T$. Let $< \mathbf{x}, \mathbf{y} >= \mathrm{Re}\{\mathbf{x}^*\mathbf{y}\}$, for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, $n > 0$, where * is the conjugate transpose. The notation x_{kj} is used to indicate $x_k - x_j$, e.g. $\theta_{kj} = \theta_k - \theta_j$.

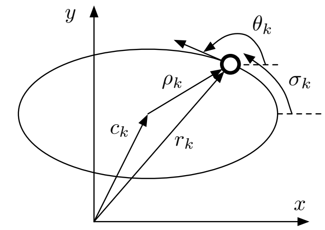

Fig. 1. The curve notation for the kth particle: the position and direction of the particle are r_k and θ_k, respectively. The curve is centered at c_k.

Consequently, using (4),

$$\frac{d\rho}{d\theta} = \frac{d\rho}{d\phi}\frac{d\phi}{d\theta} = \pm e^{i\theta}\kappa^{-1}(\phi). \quad (5)$$

We define the *curve-phase* ψ to describe a point along the curve given by

$$\psi(\phi) = \frac{2\pi}{\Omega}\sigma(\phi), \quad (6)$$

where $\Omega = \sigma(2\pi) > 0$ is the perimeter of the curve [9]. Using (4) and (6), we obtain the *curve-phase model*,

$$\dot{\psi} = \frac{d\psi}{dt} = \frac{2\pi}{\Omega}\frac{d\sigma}{d\theta}\frac{d\theta}{dt} = \pm\frac{2\pi}{\Omega}\kappa^{-1}(\phi)\dot{\theta}. \quad (7)$$

Let $\rho_k = \rho(\phi(\theta_k))$, $\kappa_k = \kappa(\phi(\theta_k))$, and $\psi_k = \psi(\phi(\theta_k))$ for $k \in \{1, \ldots, N\}$. If curve C satisfies the velocity constraint for particle k, then the center of the curve is

$$c_k = r_k \mp \rho_k. \quad (8)$$

Using (5), along solutions of the particle model (1), the center moves according to

$$\dot{c}_k = e^{i\theta_k}(1 - \kappa_k^{-1}u_k). \quad (9)$$

The curve notation is shown in Figure 1 for a particle traveling counter-clockwise around an ellipse.

Proposition 1: Under the control $u_k = \kappa_k$, each particle travels on curve C with fixed center c_k. When all the centers coincide, this corresponds to $\mathbf{c} = c_0\mathbf{1}$, $c_0 \in \mathbb{C}$.

Proof: Using (9) with $u_k = \kappa_k$ gives $\dot{c}_k = 0$. If $c_1 = \ldots = c_N = c_0$, then $\mathbf{c} = c_0\mathbf{1}$. ■

Remark 1: Using (7) with $u_k = \kappa_k$, we obtain $\dot{\psi}_k = \frac{2\pi}{\Omega}$ for all $k \in \{1, \ldots, N\}$. Consequently, particles on the same curve remain phase-locked with respect to their curve-phases since $\dot{\psi}_{kj} = 0$. This result is intuitive since, by the constant speed assumption, the arc separation between particles on the same curve remains constant and, using (6), this implies that the relative curve-phases remain constant as well.

For illustration of the method, we consider a class of smooth, closed convex curves known as *superellipses*, which includes circles, ellipses, and rounded rectangles. A parametrization of a superellipse is

$$\rho(\phi) = a\,(\cos\phi)^{\frac{1}{p}} + ib\,(\sin\phi)^{\frac{1}{p}} \quad (10)$$

for $p = 1, 3, 5, \ldots$ and $a, b > 0$. For $a > b$ (resp. $a = b$), $p = 1$ is an ellipse (resp. circle) and $p \geq 3$ is a rounded rectangle (resp. rounded square).

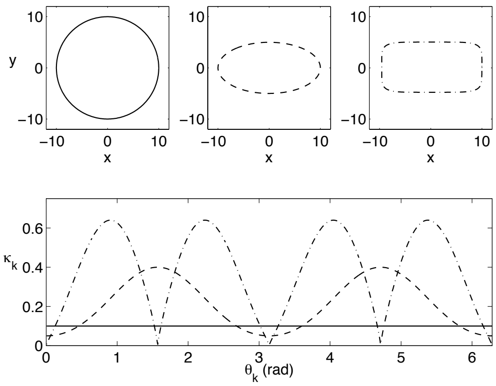

Fig. 2. Top (left to right): superellipses with $p = 1, a = b = 10$ (circle); $p = 1, a = 10, b = 5$ (ellipse); and $p = 3, a = 10, b = 5$ (rounded rectangle). Bottom: curvature, κ_k, as a function of tangent angle, θ_k, for the top panels with the corresponding line style, i.e. solid (circle), dash (ellipse) and dash-dot (rounded rectangle). Setting $\dot{\theta}_k = \kappa_k$ drives the kth particle around the corresponding curve with an arbitrary center.

By differentiating (10) and using the velocity constraint, we obtain $\tan\theta = -\frac{b}{a}(\cot\phi)^{\frac{2p-1}{p}}$ which can, in turn, be used to find $\sigma(\phi)$, ρ_k, κ_k, and ψ_k (omitted for brevity). For example, with $p = 1$, we obtain the curvature of an ellipse,

$$\kappa_k = \frac{1}{a^2b^2}\left(a^2\sin^2\theta_k + b^2\cos^2\theta_k\right)^{\frac{3}{2}}. \tag{11}$$

Setting $b = a$ in (11), one obtains the constant curvature of a circle with radius a, $\kappa_k = \frac{1}{a}$. The particle orbits and curvatures for $p = 1$ and 3 are illustrated in Figure 2 for positive ($\kappa_k > 0$) rotation[2].

Remark 2: Superellipses with $p = 1$, i.e. ellipses and circles, have definite curvature. For $p \geq 3$, as can be seen in Figure 2, the curvature κ_k is zero for $\{\theta_k \mid \theta_k = \frac{\pi}{2}j,\ j = 0, 1, 2, 3\}$ which is a set of measure zero. All superellipses are strictly convex, i.e. they are convex curves with no linear parts. The results in this paper appear (at least in simulation) to extend to superellipses with $p \geq 3$, provided that care is taken to avoid singularities in computing ψ_k.

IV. CONTROL TO SAME CURVE

In order to drive the particles to orbit the same curve, we choose a stabilizing control that minimizes the candidate Lyapunov function [12]

$$S(\mathbf{r}, \boldsymbol{\theta}) = Q(\mathbf{c}) = \frac{1}{2} < \mathbf{c}, L\mathbf{c} > . \tag{12}$$

The potential (12) is zero for $\mathbf{c} = c_0\mathbf{1}$, where $c_0 \in \mathbb{C}$, and positive otherwise. The time-derivative of $S(\mathbf{r}, \boldsymbol{\theta})$ along the solutions of (1) is

$$\dot{S}(\mathbf{r}, \boldsymbol{\theta}) = \sum_{k=1}^{N} < e^{i\theta_k}, L_k\mathbf{c} > (1 - \kappa_k^{-1}u_k),$$

where L_k denotes the kth row of the matrix L. Choosing

$$u_k = \kappa_k(1 + K_0 < e^{i\theta_k}, L_k\mathbf{c} >), \quad K_0 > 0 \tag{13}$$

results in $S(\mathbf{r}, \boldsymbol{\theta}) = -K_0\sum_{k=1}^{N} < e^{i\theta_k}, L_k\mathbf{c} >^2 \leq 0$. Note, the kth particle control depends only on the relative position and directions of the other particles to which it is connected and so the closed-loop system is invariant to rigid translation of all particles. Lyapunov analysis provides the following.

Theorem 1: All solutions of the particle model (1) with control (13) converge to the set of trajectories in which each particle orbits curve C centered at c_0.

Proof: The potential $S(\mathbf{r}, \boldsymbol{\theta})$ is positive definite and proper in the $2N - 2$ dimensional (reduced) space of the relative positions of the curve centers. Since $S(\mathbf{r}, \boldsymbol{\theta})$ is nonincreasing, by the LaSalle Invariance principle, solutions in the reduced space converge to the largest invariant set where $< e^{i\theta_k}, L_k\mathbf{c} > \equiv 0$ for $k = 1, \ldots, N$. In this set, $\dot{\theta}_k = \kappa_k$ and c_k is constant, which means the invariance condition holds only if $L\mathbf{c} \equiv 0$, i.e. $\mathbf{c} = c_0\mathbf{1}$, where $c_0 \in \mathbb{C}$ is determined by initial conditions. The conclusion follows from Proposition 1. ■

V. CURVE-PHASE CONTROL

In previous work [3], we designed gradient controls of a phase potential to stabilize symmetric patterns of the particle phases on circular orbits. For the more general class of curves proposed here, we again design gradient controls of a phase potential, but instead of the particle phases, $\boldsymbol{\theta}$, we use the particle curve-phases, $\boldsymbol{\psi}$, defined in (6). In this section and its sequel, we consider the curve-phase model (7) with control $\dot{\theta}_k = u_k = u_k(\boldsymbol{\psi})$, $\boldsymbol{\psi} \in \mathbb{T}^N$.

In order to control the particle relative curve-phase, we use phase potentials of the form [12]

$$W_1(\boldsymbol{\psi}) = Q(e^{i\boldsymbol{\psi}}) = \frac{1}{2} < e^{i\boldsymbol{\psi}}, Le^{i\boldsymbol{\psi}} > . \tag{14}$$

Note that the phase potential (14) is zero for $\boldsymbol{\psi}$ *synchronized*, i.e. $e^{i\boldsymbol{\psi}} = e^{i\psi_0}\mathbf{1}$, where $\psi_0 \in \mathbb{T}$. The gradient of $W_1(\boldsymbol{\psi})$ is

$$\frac{\partial W_1}{\partial \psi_k} = < ie^{i\psi_k}, L_k e^{i\boldsymbol{\psi}} >,\ k = 1, \ldots, N. \tag{15}$$

Choosing the (gradient) control law

$$u_k = \kappa_k\left(1 + K_1\frac{\partial W_1}{\partial \psi_k}\right), \quad K_1 \neq 0 \tag{16}$$

results in

$$\dot{W}_1(\boldsymbol{\psi}) = K_1\frac{2\pi}{\Omega}\sum_{k=1}^{N} < ie^{i\psi_k}, L_k e^{i\boldsymbol{\psi}} >^2, \tag{17}$$

where we used Property P3 to obtain

$$\sum_{k=1}^{N} < ie^{i\psi_k}, L_k e^{i\boldsymbol{\psi}} > = < i(B^T e^{i\boldsymbol{\psi}}), B^T e^{i\boldsymbol{\psi}} > = 0. \tag{18}$$

We show by Lyapunov analysis in Theorem 2 below that all solutions of the curve-phase model (7) with the control (16) converge to the set of critical points of $W_1(\boldsymbol{\psi})$, i.e. the set of configurations for which $\frac{\partial W_1}{\partial \boldsymbol{\psi}} = 0$. The following proposition partially characterizes these critical points.

Proposition 2: [12] Let L be the Laplacian matrix of a connected graph.

[2] Without loss of generality, we assume $\kappa_k > 0$ in the rest of the paper.

i. If $\bar{\boldsymbol{\psi}}$ is a critical point of the phase potential $W_1(\boldsymbol{\psi})$ then there exists a nonnegative, real vector $\boldsymbol{\alpha} \in \mathbb{R}^N$ such that

$$(L - \mathrm{diag}(\boldsymbol{\alpha}))e^{i\bar{\boldsymbol{\psi}}} = 0. \tag{19}$$

ii. If $e^{i\bar{\boldsymbol{\psi}}}$ is an eigenvector of L (resp. A) with eigenvalue λ (resp. η) then $\bar{\boldsymbol{\psi}}$ is a critical point of $W_1(\boldsymbol{\psi})$ with $\boldsymbol{\alpha} = \lambda\mathbf{1}$ (resp. $\boldsymbol{\alpha} = \boldsymbol{d} - \eta\mathbf{1}$).

iii. The Hessian of $W_1(\boldsymbol{\psi})$ evaluated at a critical point, $\bar{\boldsymbol{\psi}}$, is given by the weighted Laplacian,

$$H_1(\bar{\boldsymbol{\psi}}) = B\Phi(\bar{\boldsymbol{\psi}})B^T, \tag{20}$$

where the weight matrix is defined by

$$\Phi(\bar{\boldsymbol{\psi}}) = \mathrm{diag}(\cos(B^T\bar{\boldsymbol{\psi}})) \in \mathbb{R}^{e\times e}. \tag{21}$$

Proof: (i) Using (15), we observe that $\bar{\boldsymbol{\psi}}$ is a critical point of $W_1(\boldsymbol{\psi})$ if $L_k e^{i\bar{\boldsymbol{\psi}}} = \alpha_k e^{i\bar{\psi}_k}$, for $k = 1, \ldots, N$, which is equivalent to (19). (ii) If $Le^{i\bar{\boldsymbol{\psi}}} = \lambda e^{i\bar{\boldsymbol{\psi}}}$ (resp. $Ae^{i\bar{\boldsymbol{\psi}}} = \eta e^{i\bar{\boldsymbol{\psi}}}$) then (19) can be solved for $\boldsymbol{\alpha}$. (iii) The Hessian of $W_1(\boldsymbol{\theta})$ is determined by

$$\frac{\partial^2 W_1}{\partial \psi_k^2} = d_k - < e^{i\psi_k}, L_k e^{i\boldsymbol{\psi}} > = \sum_{j\in\mathcal{N}_k} < e^{i\psi_k}, e^{i\psi_j} > \tag{22}$$

and, for $j \neq k$,

$$\frac{\partial^2 W_1}{\partial \psi_j \partial \psi_k} = \begin{cases} - < e^{i\psi_k}, e^{i\psi_j} >, & \text{if } j \in \mathcal{N}_k, \\ 0, & \text{otherwise.} \end{cases} \tag{23}$$

Equations (22) and (23) for $H_1(\bar{\boldsymbol{\psi}})$ are equivalent to the weighted Laplacian (20). Let $f \in \{1, \ldots, e\}$ be the index of the edge connecting j and k. The corresponding weight is $\Phi_{ff} =< e^{i\bar{\theta}_k}, e^{i\bar{\theta}_j} >= \cos\bar{\theta}_{kj}$ in agreement with (21). ■

Theorem 2: All solutions of the curve-phase model (7) with control (16) converge to the set in which the curve-phase arrangement, $\boldsymbol{\psi} \in \mathbb{T}^N$, is a critical point of $W_1(\boldsymbol{\psi})$. If $K_1\Phi(\bar{\boldsymbol{\psi}}) < 0$ (resp. $K_1\Phi(\bar{\boldsymbol{\psi}}) > 0$) then $\bar{\boldsymbol{\psi}}$ is an asymptotically stable (resp. unstable) critical point that is isolated in the shape manifold $\mathbb{T}^N/\mathbb{T}$.

Proof: The potential $W_1(\boldsymbol{\psi})$ is positive definite in the $N-1$ dimensional space of relative curve-phases, $\mathbb{T}^N/\mathbb{T}$. From (17), the evolution of the phase potential $W_1(\boldsymbol{\psi})$ is monotonic along the solutions of (7). In particular, $W_1(\boldsymbol{\psi})$ is nonincreasing (nondecreasing) for $K_1 < 0$ ($K_1 > 0$). Using the LaSalle Invariance principle, solutions converge to the largest invariant set Λ where $< ie^{i\psi_k}, L_k e^{i\boldsymbol{\psi}} >\equiv 0$ for $k = 1, \ldots, N$. In this set, $\dot{\theta}_k = \kappa_k$ and $\dot{\psi}_k = \frac{2\pi}{\Omega}$. Using (15), we see that the invariance condition is satisfied at the critical points of $W_1(\boldsymbol{\psi})$.

Using Proposition 2, if $L = BB^T$ is the Laplacian of a connected graph and $\Phi(\bar{\boldsymbol{\psi}})$ is definite, then $H_1(\bar{\boldsymbol{\psi}})$ has rank $N-1$ with the zero eigenvector $\mathbf{1}$. The stability result follows from the fact that the Jacobian of the control (16) is equal to the Hessian of the phase potential. The condition $\Phi(\bar{\boldsymbol{\psi}}) < 0$ (resp. $\Phi(\bar{\boldsymbol{\psi}}) > 0$) implies that all eigenvalues of $H_1(\bar{\boldsymbol{\psi}})$ other than the simple zero eigenvalue are negative (resp. positive) and the critical point is isolated in the $\mathbb{T}^N/\mathbb{T}$. ■

Remark 3: By Proposition 1, in the particle model (1) with the control (16), each particle also converges to curve C centered at c_k.

VI. ISOLATING SYMMETRIC PATTERNS

In this section, we provide control laws to isolate a particular set of critical points of the curve-phase potential $W_1(\boldsymbol{\psi})$ composed of symmetric patterns. As in the previous section, we consider only the curve-phase model (7) with control $u_k = u_k(\boldsymbol{\psi})$.

A *symmetric* (M, N)*-pattern* is a curve-phase arrangement that has M clusters of $\frac{N}{M}$ synchronized curve-phases, where $M \in \mathrm{div}N$ [3]. The curve-phase of cluster l is

$$\Psi_l = \frac{2\pi}{M}(l-1) + \Psi_0, \tag{24}$$

for $l = 1, \ldots, M$ and $\Psi_0 \in \mathbb{T}$. For any $N \geq 2$ curve-phases, it is always possible to form at least two (M, N)-patterns: the synchronized $(1, N)$-pattern and the *splay state* (N, N)-pattern. The splay state is an arrangement in which the curve-phases are uniformly distributed around the unit circle with curve-phase differences equal to multiples of $\frac{2\pi}{N}$. If $e^{i\boldsymbol{\psi}}$ is an (M, N)-pattern with $M \neq 1$, then $\boldsymbol{\psi}$ is also *balanced*, i.e. $\mathbf{1}^T e^{i\boldsymbol{\psi}} = 0$ by definition. In fact, $\boldsymbol{\psi}$ satisfies $\mathbf{1}^T e^{im\boldsymbol{\psi}} = 0$ for $m = 1, \ldots, M-1$ and $\mathbf{1}^T e^{iM\boldsymbol{\psi}} = N$ [3].

In anticipation of our goal to isolate symmetric patterns of curve-phases, we restrict the interconnection topology to d_0-circulant graphs. All d_0-circulant graphs are d_0-regular, which means that $d_k = d_0$ for all k. Both the adjacency and Laplacian matrices of a circulant graph are circulant, i.e. they are completely defined by their first row. Each subsequent row of a circulant matrix is the previous row shifted one position to the right with the first element equal to the last element of the previous row. For example, the complete graph (all-to-all) is $(N-1)$-circulant and the cyclic graph (ring topology) is 2-circulant.

Every circulant matrix can be diagonalized by the discrete (inverse) Fourier Transform matrix, F, which is a unitary matrix with components $[F]_{k,j} = \frac{1}{\sqrt{N}} e^{i\frac{2\pi}{N}(k-1)(j-1)}$, $j, k = 1, \ldots, N$ [13]. The eigenvectors of any circulant matrix can be chosen from the columns of F.

Proposition 3: The curve-phase arrangement $\boldsymbol{\psi} \in \mathbb{T}^N$ is a symmetric pattern if and only if $e^{i\boldsymbol{\psi}}$ is a column of F.

Proof: (If) Let $e^{i\boldsymbol{\psi}}$ be the jth column of F, with $j \neq 1$. Note the first column of F is $\mathbf{1}$ which corresponds to the $(1, N)$-pattern. Because F is unitary, we observe that $\boldsymbol{\psi}$ is balanced since $\mathbf{1}^T e^{i\boldsymbol{\psi}} = 0$. We see that $\psi_k = \frac{2\pi}{N}(j-1)(k-1)$, $k = 1, \ldots, N$. If $j - 1 \in \mathrm{div}N$, then let $M = \frac{N}{j-1}$; otherwise, let $M = N$. In the case $M \neq N$, using (24) with $\Psi_0 = 0$, each curve-phase $\psi_k = \frac{2\pi}{M}(k-1)$ lies in one of M balanced clusters. In the case $M = N$, $\boldsymbol{\psi}$ is the splay state. The latter claim is proven by contradiction. Suppose there exists a cluster of size two or more that includes curve-phases ψ_k and ψ_l. Then $\psi_k + 2\pi n = \psi_l$ for $n \in \mathbb{Z}$. Using $\psi_k = \frac{2\pi}{N}(j-1)(k-1)$, we obtain $n = \frac{j-1}{N}(l-k)$ which is a contradiction since $n \notin \mathbb{Z}$ because $|l-k| < N$ and $j - 1 \notin \mathrm{div}N$.

(Only if) Let $\boldsymbol{\psi}$ be an (M, N)-pattern with cluster phases given by (24) with $\Psi_0 = 0$. If $M > 1$, letting $\psi_k = \frac{2\pi}{N}(j-1)(k-1)$, we find that $e^{i\boldsymbol{\psi}}$ is the jth column of F with $j = 1 + \frac{N}{M}$. If $M = 1$, then $e^{i\boldsymbol{\psi}}$ is the first column of F. ■

Remark 4: Using Theorem 2 and Proposition 3, we observe that symmetric patterns are critical points of the curve-phase potential $W_1(\boldsymbol{\psi})$ if L is a circulant graph Laplacian.

As in previous work, phase potentials with higher harmonics are used to isolate these symmetric patterns [3]. In order to control higher harmonics of the particle relative curve-phase, we use

$$W_m(\boldsymbol{\psi}) = Q(\frac{1}{m}e^{im\boldsymbol{\psi}}) = \frac{1}{2m^2} < e^{im\boldsymbol{\psi}}, Le^{im\boldsymbol{\psi}} >, \quad (25)$$

where $m \in \mathbb{N}$.

Let $M \in \mathrm{div}N$. Consider the curve-phase potential

$$U^{M,N}(\boldsymbol{\psi}) = \sum_{m=1}^{M} K_m W_m(\boldsymbol{\psi}), \quad (26)$$

where $W_m(\boldsymbol{\psi})$ is given by (25) and $K_m > 0$, for $m = 1, \ldots, M-1$ and $K_M < 0$. A (gradient) control is

$$u_k = \kappa_k \left(1 + \frac{\partial U^{M,N}}{\partial \psi_k}\right), \quad (27)$$

where $K_m > 0$, for $m = 1, \ldots, M-1$ and $K_M < 0$. Lyapunov analysis gives the following result.

Theorem 3: In the curve-phase model (7) under the control (27), the set of symmetric (M, N)-patterns are asymptotically stable if

$$|K_M| > M \sum_{m=1}^{M-1} \frac{K_m}{m} > 0. \quad (28)$$

Proof: The time-derivative of $U^{M,N}(\boldsymbol{\psi})$ along solutions of (7) under the control (27) is

$$\dot{U}^{M,N}(\boldsymbol{\psi}) = \frac{2\pi}{\Omega} \sum_{k=1}^{N} \sum_{m=1}^{M} \frac{K_m}{m^2} < ie^{im\psi_k}, L_k e^{im\boldsymbol{\psi}} >^2.$$

Using the compactness of $\mathbb{T}^N$, we apply the LaSalle Invariance principle to find that solutions converge to the largest invariant set Λ where $< ie^{im\psi_k}, L_k e^{im\boldsymbol{\psi}} > \equiv 0$ for $k = 1, \ldots, N$ and $m = 1, \ldots, M$. In this set, which is the set of critical points of (26), $\dot{\theta}_k = \kappa_k$ and $\dot{\psi}_k = \frac{2\pi}{\Omega}$. Next, we show that every symmetric pattern is a critical point.

If $\bar{\boldsymbol{\psi}}$ is a symmetric (M, N)-pattern, then, by Proposition 3, $e^{i\bar{\boldsymbol{\psi}}}$ is also an eigenvector of L. To prove that $\boldsymbol{\psi} \in \Lambda$ we show that, under multiplication by m, $\boldsymbol{\psi}$ becomes a new symmetric pattern with $\tilde{M}$ clusters and, therefore, $e^{im\bar{\boldsymbol{\psi}}}$ is also eigenvalue of L. Using (24), the new cluster locations are $m\Psi_l = \frac{2\pi}{M} m(l-1)$. If $m \in \mathrm{divM}$ then $\tilde{M} = \frac{M}{m}$. Otherwise, $\tilde{M} = M$ and no two clusters are synchronized. We prove this, as in the proof of Proposition 3, by contradiction. Suppose two clusters, l and k, are synchronized after their phases are multiplied by m. Then $\frac{2\pi}{M} m(l-1) + 2\pi n = \frac{2\pi}{M} m(k-1)$, where $n \in \mathbb{Z}$. However, this gives a contradiction since $n = \frac{m}{M}(k-l) \notin \mathbb{Z}$ because $\frac{m}{M} \notin \mathbb{Z}$ and $|k-l| < M$.

To show that symmetric patterns are stable critical points, we follow the same procedure as in Proposition 2. The Hessian of $W_m(\bar{\boldsymbol{\psi}})$ is given by the weighted Laplacian $H_m(\bar{\boldsymbol{\psi}}) = B\Phi(m\bar{\boldsymbol{\psi}})B^T$, where the weight matrix is defined by (21). The Hessian of $U^{M,N}(\boldsymbol{\psi})$ evaluated at $\bar{\boldsymbol{\psi}}$ is

$$H^{M,N}(\bar{\boldsymbol{\psi}}) = B \left(\sum_{k=1}^{M} \frac{K_m}{m} \Phi(m\bar{\boldsymbol{\psi}})\right) B^T, \quad (29)$$

where $K_m > 0$ for $m = 1, \ldots, M-1$, and $K_M < 0$. As in Theorem 2, if $L = BB^T$ is the Laplacian of a connected graph and $\Phi^{M,N}(\bar{\boldsymbol{\psi}}) = \sum_{m=1}^{M} \frac{K_m}{m} \Phi(m\bar{\boldsymbol{\psi}})$ is definite, then $H^{M,N}(\bar{\boldsymbol{\psi}})$ has rank $N-1$ with the zero eigenvector $\mathbf{1}$. Therefore, $\bar{\boldsymbol{\psi}}$ is isolated in the shape space $\mathbb{T}^N/\mathbb{T}$ and is stable if $\Phi^{M,N}(\bar{\boldsymbol{\psi}})$ is negative definite.

To complete the proof, we need to find conditions on the gains K_m, $m = 1, \ldots, M$, to ensure that $\Phi^{M,N}(\bar{\boldsymbol{\psi}})$ is negative definite. For L circulant, it suffices to check that the diagonal components of $\Phi^{M,N}(\bar{\boldsymbol{\psi}})$ corresponding to edges associated to a single vertex k are negative. Without loss of generality, we choose $k = 1$ and $\bar{\psi}_1 = 0$. Using (21) and (29), we obtain the stability condition,

$$\sum_{m=1}^{M} \frac{K_m}{m} \cos(\frac{2\pi}{M} mj) < 0, \ \forall\, j \in \mathcal{N}_1. \quad (30)$$

Choosing the gain $K_M < 0$ according to (28) ensures that condition (30) is satisfied, which concludes the proof. ■

Remark 5: Simulations of the curve-phase model (7) with the control (27) suggest that the choice of gain K_M given in (28) is conservative. For example, simulations show local convergence to the desired (M, N)-pattern for $|K_m| = K$, $m = 1, \ldots, M$, which does not satisfy (28). In addition, for $M = N$, simulations show convergence to the splay state for $K_m = 0$ for $m > \lfloor \frac{N}{2} \rfloor$, which is the largest integer less than or equal to $\frac{N}{2}$.

VII. STABILIZING SYMMETRIC FORMATIONS

The particles are arranged in a *symmetric* (M, N)-*formation* if each particle orbits a curve C centered at c_k and the curve-phase arrangement $\boldsymbol{\psi}$ is an (M, N)-pattern. To stabilize a symmetric formation with $\mathbf{c} = c_0\mathbf{1}$, $c_0 \in \mathbb{C}$, we choose a stabilizing control that minimizes the potential

$$V(\mathbf{r}, \boldsymbol{\theta}) = K_0 Q(\mathbf{c}) + \frac{\Omega}{2\pi} U^{M,N}(\boldsymbol{\psi}), \ K_0 > 0, \quad (31)$$

where the spacing potential $Q(\mathbf{c})$ is given by (12), the phase potential $U^{M,N}(\boldsymbol{\psi})$ is given by (26). Choosing the control

$$u_k = \kappa_k \left(1 + K_0 < e^{i\theta_k}, L_k\mathbf{c} > + \frac{\partial U^{M,N}}{\partial \psi_k}\right) \quad (32)$$

results in $\dot{V} \leq 0$. Lyapunov analysis yields the following.

Theorem 4: Let $M \in \mathrm{div}N$. In the particle model (1) with the control (32) that satisfies (28), the set of symmetric (M, N)-formations in which each particle orbits curve C centered at $c_0 \in \mathbb{C}$ is asymptotically stable.

Proof: The potential $V(\mathbf{r}, \boldsymbol{\theta})$ given in (31) is positive definite in the $3N-3$ dimensional (reduced) space consisting of the relative positions of the curve centers and the relative

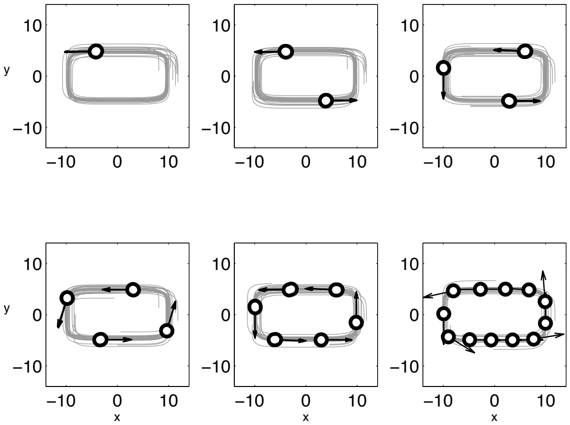

Fig. 3. Numerical simulations of the control (34) with $K_0 = 0.1$ and beacon locations $\mathbf{b} = 0$ for a 4-circulant graph and random, local initial conditions. The simulation time is 8Ω, or approximately eight revolutions of the superellipse. The panels show the six symmetric patterns for $N = 12$ on a superellipse with $p = 3$ and $M = 1, 2, 3, 4, 6$, and 12 clusters. For each $M < N$, the gains are $K_m = 0.1$ for $m = 1, \ldots, M-1$ and $K_M = -0.1$. For $M = N = 12$, the gains are $K_m = 0.1$ for $m = 1, \ldots, 6$ and $K_m = 0$ for $m > 6$. The steady-state curve-phase differences between the clusters in each simulation are equal to $\frac{2\pi}{M}$.

curve-phases. Since $V(\boldsymbol{r}, \boldsymbol{\theta})$ is nonincreasing along the solutions, by the LaSalle Invariance principle, solutions in the reduced space converge to the largest invariant set Λ where

$$< e^{i\theta_k}, L_k \mathbf{c} > + \sum_{m=1}^{M} \frac{K_m}{m} < ie^{im\psi_k}, L_k e^{im\boldsymbol{\psi}} > \equiv 0 \quad (33)$$

for $k = 1, \ldots, N$. In this set, $\dot{\theta}_k = \kappa_k$, $\dot{\psi}_k = \frac{2\pi}{\Omega}$ and c_k is constant for all $k = 1, \ldots N$. It is straightforward to show that $\frac{d}{dt} < ie^{im\psi_k}, L_k e^{im\boldsymbol{\psi}} >= 0$ for any $m \in \mathbb{N}$ and $\dot{\boldsymbol{\psi}} = \frac{2\pi}{\Omega}\mathbf{1}$. Therefore, differentiating (33) with respect to time in Λ gives $< ie^{i\theta_k}, L_k \mathbf{c} > \kappa_k \equiv 0$ for $k = 1, \ldots, N$ which can hold only if $L\mathbf{c} \equiv 0$, i.e., all particles orbit the same fixed curve. Thus, the invariance condition (33) becomes $\sum_{m=1}^{M} \frac{K_m}{m} < ie^{im\psi_k}, L_k e^{im\boldsymbol{\psi}} > \equiv 0$. Using Theorem 3, if $\boldsymbol{\psi}$ is a symmetric (M, N)-pattern, then it is an asymptotically stable point in the set Λ. ■

Simulations suggest a large region of attraction for each (M, N)-formation for the complete graph but not necessarily for d_0-circulant graphs with $d_0 < N - 1$. To demonstrate convergence of the closed-loop system with limited communication, we select initial conditions near the desired (M, N)-formation. The six symmetric patterns for $N = 12$ on a superellipse with $p = 3$ are shown in Figure 3 for a 4-circulant graph.

VIII. EXTENSIONS AND ONGOING WORK

In applications that use these control strategies to coordinate a group of autonomous vehicles, it is useful to consider several extensions of the particle model (1) and the symmetric formation control (32). For example, a useful extension is to stabilize symmetric formations with prescribed curve centers using a model with constant non-unit speed particles. Let $s \in \mathbb{R}^+$ be the (constant) particle speed and let $\mathbf{b} \in \mathbb{C}^N$ denote the vector of desired curve centers or beacons. To drive the particles to orbit these beacons in an (M, N)-pattern, we replace the potential $V(\boldsymbol{r}, \boldsymbol{\theta})$ with

$$\hat{V}(\boldsymbol{r}, \boldsymbol{\theta}) = \frac{K_0}{2}\|\mathbf{c} - \mathbf{b}\|^2 + \frac{\Omega}{2\pi} U^{M,N}(\boldsymbol{\psi}), \; K_0 > 0,$$

and the control u_k with

$$u_k = \kappa_k \left(s + K_0 < e^{i\theta_k}, c_k - b_k > + \frac{\partial U^{M,N}}{\partial \psi_k} \right). \quad (34)$$

In this way, one can obtain a local convergence result analogous to Theorem 4 (omitted for brevity) which is illustrated in Figure 3 for $\mathbf{b} = 0$.

In this paper, we provide feedback control laws to stabilize collective motion of a group of planar particles. In ongoing work, we seek to address limitations of the approach presented here such as the restriction to curves with definite curvature. In addition, we seek to remove the restriction to fixed, connected, and undirected interconnection graphs which is an unrealistic assumption for many mobile sensor networks. This restriction is removed in [14], which extends the design of symmetric patterns on circular orbits to time-varying and/or directed graphs.

IX. ACKNOWLEDGMENTS

The authors gratefully acknowledge discussions with Fumin Zhang regarding this work.

REFERENCES

[1] N. E. Leonard, D. A. Paley, F. Lekien, R. Sepulchre, D. M. Fratantoni, and R. E. Davis, "Collective motion, sensor networks and ocean sampling," *Proc. of the IEEE*, in press.

[2] E. W. Justh and P. S. Krishnaprasad, "A simple control law for UAV formation flying," Univ. Maryland, Tech. Rep. 2002-38, 2002.

[3] R. Sepulchre, D. A. Paley, and N. E. Leonard, "Stabilization of planar collective motion: all-to-all communication," to appear in *IEEE Trans. Autom. Control*.

[4] E. W. Justh and P. S. Krishnaprasad, "Equilibria and steering laws for planar formations," *Sys. Cont. Lett.*, vol. 52, no. 1, pp. 25–38, 2004.

[5] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Autom. Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[6] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[7] J. A. Marshall and M. E. Broucke, "On invariance of cyclic group symmetries in multiagent formations," in *Proc. Joint 44th IEEE CDC and ECC*, December 2005, pp. 746–751.

[8] J. Rogge and D. Aeyels, "Decentralized control of vehicle platoons with interconnection possessing ring topology," in *Proc. Joint 44th IEEE CDC and ECC*, December 2005, pp. 1491–1496.

[9] F. Zhang, E. Justh, and P. S. Krishnaprasad, "Boundary following using gyroscopic control," in *Proc. 43rd IEEE CDC*, December 2004, pp. 5204–5209.

[10] F. Zhang and N. E. Leonard, "Coordinated patterns on smooth curves," in *Proc. IEEE Int. Conf. Net., Sens. Cont.*, April 2006, pp. 434–440.

[11] C. Godsil and G. Royle, *Algebraic Graph Theory*, ser. Graduate Texts in Mathematics. Springer-Verlag, 2001, no. 207.

[12] R. Sepulchre, D. Paley, and N. E. Leonard, "Graph Laplacian and Lyapunov design of collective planar motions," in *Proc. Int. Symp. NOLTA*, October 2005.

[13] P. J. Davis, *Circulant Matrices*. John Wiley & Sons, Inc., 1979.

[14] R. Sepulchre, D. A. Paley, and N. E. Leonard, "Stabilization of planar collective motion with limited communication," submitted to *IEEE Trans. Autom. Control*.