Optimal guidance of a team of mobile actuators for controlling a 1D diffusion process with unknown initial conditions

Sheng Cheng and Derek A. Paley

***Abstract*— This paper proposes an optimization framework for steering a team of mobile actuators to control a diffusion process with unknown initial conditions. The optimization problem seeks a guidance strategy that minimizes the quadratic cost of controlling the diffusion process subject to the worst possible initial condition and the quadratic cost of steering the mobile actuators. We turn the problem into an unconstrained optimization and use a gradient-descent method to solve it. The solution of the proposed problem is suboptimal for the same problem with a known initial condition—even for the worst-case initial condition. This suboptimality property suggests the guidance strategy can be implemented when the initial condition of the diffusion process is unknown. A numerical example compares solutions with known and unknown initial conditions.**

I. INTRODUCTION

Monitoring and containing large-scale spatiotemporal processes like forest fires, oil spills, and harmful algal blooms can be challenging. Potential consequences include environmental damage, economic losses, and even health threats to human operators. A spatiotemporal process varies both in space and time, which can be modeled by a partial differential equation (PDE), also known as distributed parameter systems (DPS). With a dynamical system model of the process, autonomous aerial/ground/surface/underwater vehicles can be deployed to estimate and/or control it. Autonomous vehicles equipped with sensors and actuators can efficiently and cooperatively complete estimation and control tasks when they are given suitable guidance.

We propose an optimization framework that guides a team of mobile actuators to control a DPS. As a representative physical model, we consider a 1D diffusion process. The optimal strategy minimizes the sum of two terms: the quadratic PDE cost of controlling the diffusion to a zero state subject to the *worst* initial condition and the quadratic mobility cost of steering the mobile actuators. Hence, the optimal guidance minimizes both the PDE cost and the mobility cost, even when the process's initial condition is unknown.

The proposed formulation is well suited for autonomous vehicles that have limited onboard resources. The problem has a finite horizon that accounts for limited battery life or fuel, which does not fit into a framework with unlimited time, such as infinite-horizon optimization or Lyapunov-based methods. The mobility cost can be considered an inequality constraint that augments the cost function with Lagrangian multipliers. Hence, the proposed problem is an intermediate step when there is an explicit constraint on mobility, such as an upper-bounded total guidance effort.

The control of a DPS can be categorized by the mobility of the actuators: stationary or mobile. The same category applies to estimation and parameter identification by sensors, but we focus on actuator control for brevity. Stationary actuators can be placed on the boundary of the spatial domain [1], where the boundary condition is specified as the control input. The actuators can also be placed within the spatial domain, where the actuation is specified as a nonhomogeneous term in the dynamics of a DPS. The problem of determining the locations of stationary actuators is called the actuator placement problem. Various optimization criteria have been studied for actuator placement, for example, quadratic cost under the worst initial condition [2], [3], $\mathbb{H}_2$ [4], and maximum controllability [5]. Comparisons of different criteria have been reported for a simply supported beam [6] and for a diffusion process [7]. Maximizing the minimum eigenvalue of the controllability gramian is not a useful criterion because the lower bound of the eigenvalues of the controllability gramian is zero [6], [7].

For mobile actuators, the guidance strategy utilizes the additional degree of freedom induced from mobility to improve the control performance relative to stationary actuators. Optimal guidance in the sense of linear-quadratic (LQ) cost has been studied in [8]–[10]. One may design guidance using Lyapunov-based methods, where a Lyapunov function is constructed such that its time derivative is made negative by suitable guidance [11], [12]. Geometric methods can also be applied to guidance design, such as Centroidal Voronoi Tessellation [13], [14].

The proposed problem is closely related to a previously studied problem [10]. The previous work simultaneously solves for the control input to the DPS and the guidance of actuators to minimize the same cost function as used here, but subject to a *given* (instead of the *worst*) initial condition. We show that the solution of the proposed problem is suboptimal for this previously studied problem—even when the previous one is subject to the worst initial condition—using a max-min inequality. Hence, the optimal guidance for the proposed problem can be applied to steer the mobile actuators when the initial condition of the PDE is unknown.

The contributions of this paper are (1) formulation of the problem of steering a team of mobile actuators to control a

Work reported in this article was sponsored under a seed grant from Northrop Grumman.

S. Cheng is with the Department of Electrical and Computer Engineering and the Institute for Systems Research at the University of Maryland, College Park, MD 20742, USA. (email:{cheng@terpmail.umd.edu})

D. A. Paley is with the Department of Aerospace Engineering and the Institute for Systems Research at the University of Maryland, College Park, MD 20742, USA. (email:{dpaley@umd.edu})

diffusion process under the worst possible initial condition as an optimization problem; (2) analysis of the suboptimality of the proposed problem's solution for a previously studied problem via the max-min inequality; and (3) transformation of the proposed problem into an unconstrained optimization problem and a gradient-based solution method. Potential applications include wildfire containment, oil spill mitigation, and harmful algae control using autonomous unmanned vehicles.

The remainder of the paper is organized as follows. Section II introduces the relevant mathematical background, including representation of a diffusion process by an infinite-dimensional system, the associated LQ optimal control, and its finite-dimensional approximation. Section III formulates the optimization problem and analyzes its suboptimality for the problem studied previously. Section IV details a solution method that first turns the problem into an unconstrained optimization and then applies a gradient-descent method to solve a finite-dimensional problem approximated by the Galerkin scheme. A numerical example is provided to show the optimal trajectories and illustrate the suboptimality of the proposed problem. Section V summarizes the paper and discusses ongoing work.

II. Dynamics and modeling approximations

A. Notation and terminology

The symbol $\mathbb{R}$ denotes the set of real numbers. The boundary of a set M is denoted by ∂M. The n-nary Cartesian power of a set M is denoted by M^n. We use $|\cdot|$ and $\|\cdot\|$ for the absolute value and norm (with subscript indicating type), respectively. The superscript * denotes an optimal variable, whereas * denotes the adjoint of a linear operator. We use $V^*_{(\mathrm{P})}$ to denote the optimal cost of optimization problem (P). The transpose of matrix A is denoted by A^T. The trace of matrix C is $\mathrm{tr}\,(C)$. An $n \times n$-dimensional diagonal matrix with elements of vector $[a_1, a_2, \dots, a_n]$ on the main diagonal is denoted by $\mathrm{diag}(a_1, a_2, \dots, a_n)$. The ith row of a vector v is $[v]_i$. We adopt the following terminology [10]: *guidance* refers to steering of the mobile actuators, whereas *control* refers to actuation of the DPS.

B. Dynamics of the DPS and mobile actuators

Consider guiding m mobile actuators to control a 1D diffusion process modeled by the following PDE:

$$\frac{\partial z(x,t)}{\partial t} = a\frac{\partial^2 z(x,t)}{\partial x^2} + \sum_{i=1}^{m} b_i u_i(t) K_i(\xi_i(t), t)(x), \quad (1)$$

where $z(\cdot,\cdot)$ denotes a 1D diffusion process that has a spatial component $x \in \Omega \subset \mathbb{R}$ and a time component $t \in [0, t_f]$ for a given terminal time t_f; $u(\cdot) \in \mathcal{U} := \{u(t) \in U \subset \mathbb{R}^m,$ piecewise continuous in $t \in [0, t_f]\}$, is a (vector) function that denotes the magnitude of the actuation input; $K_i : \Omega \times [0, t_f] \to L^2(\Omega)$ characterizes how the actuation affects the diffusion process; and $\xi(t) \in \mathbb{R}^m$ is a vector of the locations of the mobile actuators. The coefficients $a \in \mathbb{R}$ and $b_i \in \mathbb{R}$ denote the diffusion coefficient and input gain of actuator i, respectively. The diffusion process has initial condition $z(\cdot, 0) = z_0(\cdot)$ and Dirichlet boundary condition $z(\cdot, t)|_{\partial\Omega} = 0$, $t \in [0, t_f]$. The following linear dynamics describe the motion of the mobile actuators:

$$\dot{\xi}(t) = \alpha\xi(t) + \beta p(t), \qquad \xi(0) = \xi_0, \quad (2)$$

where

$$\alpha = \mathrm{diag}(\alpha_1, \alpha_2, \dots, \alpha_m), \qquad \beta = \mathrm{diag}(\beta_1, \beta_2, \dots, \beta_m),$$

$\alpha_i, \beta_i \in \mathbb{R}$ for $i = 1, 2, \dots, m$, and the initial locations $\xi_0 \in \mathbb{R}^m$ are given. The guidance $p(\cdot) \in \mathcal{P} := L^2([0, t_f]; P)$ of the mobile actuators takes values in the admissible set $P \subseteq \mathbb{R}^m$.

Since PDEs can be formulated as differential equations on an abstract linear vector space of infinite dimension [15], we can compactly represent system (1) as an infinite-dimensional linear system

$$\dot{\mathcal{Z}}(t) = \mathcal{A}\mathcal{Z}(t) + \mathcal{B}(\xi(t), t)u(t), \qquad \mathcal{Z}(0) = \mathcal{Z}_0 = z_0, \quad (3)$$

where $\mathcal{Z}(t)$ belongs to a Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|_{\mathcal{H}}$. Here, the variable $\mathcal{Z}$ is the state of the DPS and space $\mathcal{H} := L^2(\Omega)$ is the state space. The operator $\mathcal{A}$ is defined as $\mathcal{A}\phi = a\partial^2\phi(x)/\partial x^2$ with $\phi \in \mathrm{Dom}(\mathcal{A}) = \{\phi \in H^2(\Omega) \cap H^1_0(\Omega)\}$. The operator $\mathcal{B}(\xi(t), t) \in \mathcal{L}(U; \mathcal{H})$ is the input operator, where $[\mathcal{B}(\xi(t), t)]_i := b_i K_i(\xi_i(t), t)$.

Assume that $\mathcal{A}$ is an infinitesimal generator of a strongly continuous semigroup $\mathcal{F}(t)$ on $\mathcal{H}$. The dynamical system (3) has a unique mild solution $\mathcal{Z}(\cdot) \in C([0, t_f]; \mathcal{H})$ for any $\mathcal{Z}_0 \in \mathcal{H}$ and any $u(\cdot) \in L^2([0, t_f]; U)$ such that

$$\mathcal{Z}(t) = \mathcal{F}(t)\mathcal{Z}_0 + \int_0^t \mathcal{F}(t-\tau)\mathcal{B}(\xi(\tau), \tau)u(\tau)\mathrm{d}\tau.$$

(Assuming state $\mathcal{Z}(t)$ is available for full-state feedback control, we do not specify an output equation. One may refer to [16] for an estimation framework with a team of mobile sensors.)

Similar to a finite-dimensional system, we can formulate a linear-quadratic regulator (LQR) associated to the differential equation (3). A general LQR minimizes the following quadratic cost:

$$\begin{aligned} J(\mathcal{Z}, u) = &\int_0^{t_f} \Big(\langle\mathcal{Z}(t), \mathcal{Q}\mathcal{Z}(t)\rangle + u(t)^T R u(t)\Big)\mathrm{d}t \\ &+ \langle\mathcal{Z}(t_f), \mathcal{Q}_f\mathcal{Z}(t_f)\rangle, \end{aligned} \quad (4)$$

where $\mathcal{Q} \in \mathcal{L}(\mathcal{H})$ and $\mathcal{Q}_f \in \mathcal{L}(\mathcal{H})$ are self-adjoint, nonnegative, Hilbert-Schmidt operators that evaluate the running cost and terminal cost, respectively, of the state $\mathcal{Z}$. The coefficient R is an $m \times m$-dimensional symmetric and positive definite real matrix that weights the control effort of the DPS. The optimal feedback control associated with a given trajectory $\xi(\cdot)$ of actuators is [15]

$$u^*(t) = -R^{-1}\mathcal{B}^\star(\xi(t), t)\mathcal{S}(t)\mathcal{Z}(t), \quad (5)$$

where $\mathcal{S} \in \mathcal{L}(\mathcal{H})$ is a self-adjoint and nonnegative operator that satisfies the operator differential Riccati equation

$$\dot{\mathcal{S}}(t) = -\mathcal{A}^\star\mathcal{S}(t) - \mathcal{S}(t)\mathcal{A} - \mathcal{Q} + \mathcal{S}(t)\bar{\mathcal{B}}\bar{\mathcal{B}}^\star(\xi(t),t)\mathcal{S}(t), \tag{6}$$

$$\mathcal{S}(t_f) = \mathcal{Q}_f, \tag{7}$$

where $\bar{\mathcal{B}}\bar{\mathcal{B}}^\star(\xi(t),t)$ is short for $\mathcal{B}(\xi(t),t)R^{-1}\mathcal{B}^\star(\xi(t),t)$. Moreover, the optimal quadratic cost $J^*(\mathcal{Z}^*,u^*)$ with the optimal control u^* and the corresponding optimal state $\mathcal{Z}^*$ satisfies $J^*(\mathcal{Z}^*,u^*) = \langle \mathcal{Z}_0, \mathcal{S}(0)\mathcal{Z}_0\rangle$ [15]. The maximum quadratic cost under the worst initial condition is the operator norm of the Riccati operator $\mathcal{S}(0)$ [2], i.e.,

$$\|\mathcal{S}(0)\|_{\text{op}} = \underset{\mathcal{Z}_0\in\mathcal{H}, \|\mathcal{Z}_0\|=1}{\text{maximize}} \langle \mathcal{Z}_0, \mathcal{S}(0)\mathcal{Z}_0\rangle. \tag{8}$$

C. Approximations of the infinite-dimensional variables

Approximations to (3) and (6) permit numerical computation. Consider a finite-dimensional subspace $\mathcal{H}_N \subset \mathcal{H}$ with dimension N. The inner product and norm of $\mathcal{H}_N$ are inherited from that of $\mathcal{H}$. Let $P_N : \mathcal{H} \to \mathcal{H}_N$ denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_N$. Let $Z_N(t) \in \mathbb{R}^N$ and $S_N(t) \in \mathbb{R}^{N\times N}$ denote the finite-dimensional approximation of $\mathcal{Z}(t)$ and $\mathcal{S}(t)$, respectively, where $Z_N(t) = P_N\mathcal{Z}(t)$ and $S_N(t) = P_N\mathcal{S}(t)P_N$, $t \in [0,t_f]$. A finite-dimensional approximation of (3) is

$$\dot{Z}_N(t) = A_N Z_N(t) + B_N(\xi(t),t)u(t), \tag{9}$$

$$Z_N(0) = Z_{N,0} = P_N\mathcal{Z}_0, \tag{10}$$

where $A_N \in \mathcal{L}(\mathcal{H}_N)$ and $B_N(\xi(t),t) \in \mathcal{L}(U,\mathcal{H}_N)$ are approximations of $\mathcal{A}$ and $\mathcal{B}(\xi(t),t)$, respectively. Let $F_N(t)$ denote the semigroup generated by A_N. We make standard assumptions of the approximation [15]:

(A1) For $z \in \mathcal{H}$, the following holds uniformly in t in bounded intervals:

$$\|F_N(t)P_N z - \mathcal{F}(t)z\|_{\mathcal{H}} \to 0,$$
$$\left\|F_N^T(t)P_N z - \mathcal{F}^\star(t)z\right\|_{\mathcal{H}} \to 0.$$

(A2) For $z \in \mathcal{H}$, $u \in U$, and $x \in \Omega$, the following holds uniformly in t in bounded intervals:

$$\|B_N(x,t)u - \mathcal{B}(x,t)u\|_{\mathcal{H}} \to 0,$$
$$\left\|B_N^T(x,t)P_N z - \mathcal{B}^\star(x,t)z\right\|_{\mathbb{R}^m} \to 0.$$

(A3) The family of pairs (A_N, B_N) is uniformly exponentially stabilizable.

The finite-dimensional approximation of (6) is

$$\dot{S}_N(t) = -(A_N)^T S_N(t) - S_N(t)A_N - Q_N + S_N(t)\bar{B}_N\bar{B}_N^T(\xi(t),t)S_N(t), \tag{11}$$

$$S_N(t_f) = Q_{fN}, \tag{12}$$

where $Q_N = P_N\mathcal{Q}P_N$, $Q_{fN} = P_N\mathcal{Q}_fP_N$, and $\bar{B}_N\bar{B}_N^T(\xi(t),t)$ is short for $B_N(\xi(t),t)R^{-1}B_N^T(\xi(t),t)$. The convergence of the approximation $S_N(\cdot)$ to $\mathcal{S}(\cdot)$ can be established under suitable assumptions (see [17, Theorem 3.5]), under which the following holds as $N \to \infty$:

$$\sup_{t\in[0,t_f]} \|\mathcal{S}(t) - S_N(t)\|_{\text{op}} \to 0. \tag{13}$$

III. Optimal guidance subject to the worst initial condition

A. Problem formulation

This subsection introduces the optimal guidance of the mobile actuators such that the following cost is minimized:

$$a_{\text{IC}}^2 \|\mathcal{S}(0)\|_{\text{op}} + J_{\text{m}}(\xi,p). \tag{14}$$

Here, $\|\mathcal{S}(0)\|_{\text{op}}$ is the operator norm of the Riccati operator solved from (6) with initial condition (7). The value of this operator norm represents the maximum value of the quadratic PDE cost (4) under the worst initial condition. The coefficient a_{IC}^2 scales the operator norm for the cases with non-unit norm of the initial condition $\mathcal{Z}_0$, i.e., when $\|\mathcal{Z}_0\|_{\mathcal{H}} = a_{\text{IC}} > 0$. The mobility cost $J_{\text{m}}(\xi,p)$ represents the cost incurred from the motion of the actuator. Consider a quadratic mobility cost

$$J_{\text{m}}(\xi,p) = \int_0^{t_f} \Big(\xi(t)^T q\xi(t) + p(t)^T\gamma p(t)\Big)\text{d}t + (\xi(t_f)-\xi_f)^T q_f(\xi(t_f)-\xi_f),$$

where q and q_f are $m \times m$-dimensional symmetric and positive semidefinite matrices, respectively; γ is an $m\times m$-dimensional positive definite matrix; and $\xi_f \in \Omega^m$ is the vector of terminal locations for the mobile actuators. Such terminal locations may be user-specified in some applications.

The problem is

$$\begin{aligned} &\underset{p\in\mathcal{P}}{\text{minimize}} && a_{\text{IC}}^2 \|\mathcal{S}(0)\|_{\text{op}} + J_{\text{m}}(\xi,p) \\ &\text{subject to} && \dot{\xi}(t) = \alpha\xi(t) + \beta p(t),\ \xi(0) = \xi_0, \end{aligned} \tag{P}$$

where the optimization constraint is the dynamics of the mobile actuators. Therefore, problem (P) identifies a guidance $p(\cdot)$ whose quadratic PDE cost subject to the worst initial condition plus the mobility cost is minimum among other feasible guidance. Problem (P) is closely related to a previously studied problem [10]. In fact, the solution of (P) is suboptimal for the previous problem, even when the latter is subject to the worst possible initial condition of the PDE, as we discuss in the next subsection.

B. Suboptimality of (P)

The problem studied in [10] finds both PDE control $u(\cdot)$ and actuator guidance $p(\cdot)$ such that the sum of the quadratic PDE cost $J(\mathcal{Z},u)$ and mobility cost $J_{\text{m}}(\xi,p)$ is minimized:

$$\begin{aligned} &\min_{p\in\mathcal{P}}\min_{u\in\mathcal{U}} && J(\mathcal{Z},u) + J_{\text{m}}(\xi,p) \\ &\text{subject to} && \dot{\mathcal{Z}}(t) = \mathcal{A}\mathcal{Z}(t) + \mathcal{B}(\xi(t),t)u(t),\ \mathcal{Z}(0) = \mathcal{Z}_0, \\ &&& \dot{\xi}(t) = \alpha\xi(t) + \beta p(t),\ \xi(0) = \xi_0. \end{aligned} \tag{P1}$$

In this subsection, without loss of generality, assume the initial condition $\mathcal{Z}_0$ is scaled to have unit $\mathcal{H}$-norm, i.e., $\mathcal{Z}_0$ is taken from the unit ball $\mathcal{H}^0 := \{\mathcal{Z} \in \mathcal{H} | \|\mathcal{Z}\|_{\mathcal{H}} = 1\}$.

Problem (P1) can be written in the following form using properties of LQR [15]:

$$\begin{aligned} &\underset{p\in\mathcal{P}}{\text{minimize}} && \langle \mathcal{Z}_0, \mathcal{S}(0)\mathcal{Z}_0\rangle + J_{\mathrm{m}}(\xi,p) \\ &\text{subject to} && \dot{\xi}(t) = \alpha\xi(t) + \beta p(t),\ \xi(0)=\xi_0, \end{aligned} \tag{P2}$$

for a given $\mathcal{Z}_0 \in \mathcal{H}^0$. Note that a solution $p^*(\cdot)$ of (P2) is also a solution of (P1). Furthermore, the optimal control $u^*(\cdot)$ of (P1) can be computed by (5) with the actuator trajectory $\xi^*(\cdot)$ steered by the optimal guidance $p^*(\cdot)$. Hence, $V^*_{(\mathrm{P1})} = V^*_{(\mathrm{P2})}$.

Next, consider how bad the performance of problem (P1) or (P2) could be when $\mathcal{Z}_0 \in \mathcal{H}^0$ is unknown. In other words, we would like to solve the following problem to find the worst initial condition and the associated optimal guidance:

$$\begin{aligned} &\max_{\mathcal{Z}_0\in\mathcal{H}^0}\min_{p\in\mathcal{P}} && \langle \mathcal{Z}_0, \mathcal{S}(0)\mathcal{Z}_0\rangle + J_{\mathrm{m}}(\xi,p) \\ &\text{subject to} && \dot{\xi}(t) = \alpha\xi(t) + \beta p(t),\ \xi(0)=\xi_0. \end{aligned} \tag{P3}$$

Problem (P3) may not have a closed-form solution. In addition, any iterative numerical procedure would need to solve the inner problem (P2) or (P1) repeatedly with initial conditions in the set $\mathcal{H}^0$. Solving (P2) or (P1) is already challenging because of the high dimensionality induced in the approximation of the infinite-dimensional components and the nonlinearity of the dynamics of the approximated problem [10, Remark 6]. Therefore, we do not seek to solve (P3). Rather, we switch the order of maximization and minimization in (P3) and observe that the resulting problem is exactly problem (P) with $a_{\mathrm{IC}} = 1$ (using (8)):

$$\begin{aligned} &\min_{p\in\mathcal{P}}\max_{\mathcal{Z}_0\in\mathcal{H}^0} && \langle \mathcal{Z}_0, \mathcal{S}(0)\mathcal{Z}_0\rangle + J_{\mathrm{m}}(\xi,p) \\ &\text{subject to} && \dot{\xi}(t) = \alpha\xi(t) + \beta p(t),\ \xi(0)=\xi_0. \end{aligned} \tag{P4}$$

Hence, $V^*_{(\mathrm{P4})} = V^*_{(\mathrm{P})}$ and a solution of (P) is also a solution of (P4).

Proposition 1: Consider problems (P), (P1), and (P3). The following inequalities hold:

$$V^*_{(\mathrm{P})} \geq V^*_{(\mathrm{P3})} \geq V^*_{(\mathrm{P1})}. \tag{15}$$

Proof: We shall use the max-min inequality: for any function $f : X \times Y \to \mathbb{R}$ [18],

$$\sup_{x\in X}\inf_{y\in Y} f(x,y) \leq \inf_{y\in Y}\sup_{x\in X} f(x,y).$$

Hence, we conclude $V^*_{(\mathrm{P4})} \geq V^*_{(\mathrm{P3})}$. Since $V^*_{(\mathrm{P4})} = V^*_{(\mathrm{P})}$, $V^*_{(\mathrm{P3})} \geq V^*_{(\mathrm{P2})}$, and $V^*_{(\mathrm{P2})} = V^*_{(\mathrm{P1})}$, the desired inequalities follow. ■

Remark 1: An alternative way to show $V^*_{(\mathrm{P})} \geq V^*_{(\mathrm{P1})}$ is by plugging the optimal guidance of (P) in problem (P1)'s cost function, whose value is denoted by $\bar{V}_{(\mathrm{P1})}$. It follows that $V^*_{(\mathrm{P})} \geq \bar{V}_{(\mathrm{P1})} \geq V^*_{(\mathrm{P1})}$.

The inequalities of (15) indicate that a solution of (P) is suboptimal for (P1). The suboptimality holds even when (P1) is subject to the worst initial condition, which is (P3).

Problem (P) is an alternative problem to solve when the initial condition $\mathcal{Z}_0$ is not exactly known except for its norm. In this situation, problem (P1) cannot be solved due to the unknown value of $\mathcal{Z}_0$. Since the optimal guidance of (P) can be pre-computed given $\|\mathcal{Z}_0\|_{\mathcal{H}}$, this guidance can always be applied to steer the actuators as an initial attempt that is suboptimal for (P1).

IV. Gradient-based solution method

We propose a numerical procedure to solve problem (P). Before we proceed, note that problem (P) can be turned into an unconstrained problem, because the actuator state can be uniquely determined by the guidance through the following equation:

$$\xi(t) = e^{\alpha t}\xi_0 + \int_0^t e^{\alpha\tau}\beta p(\tau)\mathrm{d}\tau. \tag{16}$$

This equation indicates that both $\|\mathcal{S}(0)\|_{\mathrm{op}}$ and $J_{\mathrm{m}}(\xi,p)$ are mappings from $L^2([0,t_f];P)$ to $\mathbb{R}$, which allows the derivation of the Fréchet derivative of $\|\mathcal{S}(0)\|_{\mathrm{op}}$ and $J_{\mathrm{m}}(\xi,p)$ with respect to p:

$$\begin{aligned} D_p\|\mathcal{S}(0)\|_{\mathrm{op}} &= D_{\mathcal{S}(0)}\|\mathcal{S}(0)\|_{\mathrm{op}} \\ &\quad\circ D_{\bar{\mathcal{B}}\bar{\mathcal{B}}^\star}\mathcal{S}(0)\circ D_\xi\bar{\mathcal{B}}\bar{\mathcal{B}}^\star\circ D_p\xi, \end{aligned} \tag{17}$$

$$\begin{aligned} D_pJ_{\mathrm{m}}(\xi,p) &= D_\xi\int_0^{t_f}\xi(t)^Tq\xi(t)\mathrm{d}t\circ D_p\xi \\ &\quad + D_p\int_0^{t_f}p(t)^Trp(t)\mathrm{d}t \\ &\quad + D_{\xi(t_f)}\left(\xi(t_f)^Tq_f\xi(t_f)\right)\circ D_p\xi(t_f). \end{aligned} \tag{18}$$

Hence, gradient-based methods are sufficient to solve problem (P) for a local minimum.

To solve (P) numerically, we approximate the PDE components using the Galerkin method with sinusoidal basis $\{\phi_n\}_{n=1}^N$, where $\phi_n(x) = \sqrt{2}\sin(n\pi x)$, which consists of the first N eigenfunctions of $\mathcal{A}$. This approximation method satisfies the general assumptions (A1)–(A3). Now, we have the following unconstrained problem:

$$\underset{p\in\mathcal{P}}{\text{minimize}}\quad a_{\mathrm{IC}}^2\|S_N(0)\|_{\mathrm{op}} + J_{\mathrm{m}}(\xi,p). \tag{AP}$$

Correspondingly, the Fréchet derivative of $\|S_N(0)\|_{\mathrm{op}}$ with respect to p is

$$\begin{aligned} D_p\|S_N(0)\|_{\mathrm{op}} &= D_{S_N(0)}\|S_N(0)\|_{\mathrm{op}} \\ &\quad\circ D_{\bar{B}\bar{B}_N^T}S_N(0)\circ D_\xi\bar{B}\bar{B}_N^T\circ D_p\xi. \end{aligned} \tag{19}$$

Using (19), we can evaluate increments of $\|S_N(0)\|_{\mathrm{op}}$ when p has small increments (which leads to the computation of the gradient of $\|S_N(0)\|_{\mathrm{op}}$ with respect to p). Denote a small increment of p by Δp, where $p+\Delta p\in\mathcal{P}$ and the corresponding increment $\Delta\|S_N(0)\|_{\mathrm{op}}$ is

$$\Delta\|S_N(0)\|_{\mathrm{op}}(\Delta p) = \mathrm{tr}\left(vv^T\left(\Lambda_N\circ D_\xi\bar{B}\bar{B}_N^T\circ D_p\xi(\Delta p)\right)\right), \tag{20}$$

where $v\in\mathbb{R}^N$ is the eigenvector of $S_N(0)$ associated with the maximum eigenvalue and $\Lambda_N\in\mathbb{R}^{N\times N}$ is the finite-dimensional approximation of $\Lambda := D_{\bar{\mathcal{B}}\bar{\mathcal{B}}^\star}\mathcal{S}(0)$ such that for $h\in\mathcal{L}(C([0,t_f];J_1(\mathcal{H})))$ [19]

$$\begin{aligned} \Lambda h = -\int_0^{t_f}\mathcal{F}^\star(\tau)\Big(&(\Lambda h)(\tau)\bar{\mathcal{B}}\bar{\mathcal{B}}^\star(\xi(\tau),\tau)\mathcal{S}(\tau) \\ &+\mathcal{S}(\tau)\bar{\mathcal{B}}\bar{\mathcal{B}}^\star(\xi(\tau),\tau)(\Lambda h)(\tau) \\ &+\mathcal{S}(\tau)(\Lambda h)(\tau)\mathcal{S}(\tau)\Big)\mathcal{F}(\tau)\mathrm{d}\tau. \end{aligned} \tag{21}$$

Remark 2: For a symmetric matrix $S \in \mathbb{R}^{n\times n}$, since $\|S\|_{\text{op}}$ equals the maximum eigenvalue of S, the (Fréchet) derivative of $\|S\|_{\text{op}}$ with respect to S is

$$\frac{\mathrm{d}\,\|S\|_{\text{op}}}{\mathrm{d}S} = v_{\max} v_{\max}^T, \tag{22}$$

where $v_{\max} \in \mathbb{R}^n$ is the eigenvector of S associated with the maximum eigenvalue. Note that (22) holds if the maximum eigenvalue of S is simple, which we assume here.

To utilize the Fréchet derivatives (18) and (19) in the gradient-based method, we choose a basis that spans the space of guidance functions. Let $T_f := t_f/\Delta t$ for a small time interval Δt and $\Psi_{\mathcal{P}} \subset L^2([0,t_f],\mathbb{R})$ be a set of basis functions $\{\psi_n\}_{n=1}^{T_f}$ such that

$$\psi_n(t) = \begin{cases} 1 & t \in [(n-1)\Delta t, n\Delta t) \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

The set $\Psi_{\mathcal{P}}$ characterizes an approximation of the guidance function by time discretization such that the guidance takes constant values for each interval with length Δt. Choose Δt to be sufficiently small to mimic continuous time. Let the gradient of $\|S_N(0)\|_{\text{op}}$ with respect to p on the space spanned by the basis functions in $\Psi_{\mathcal{P}}$ be a T_f-dimensional vector, denoted by $\nabla_p \|S_N(0)\|_{\text{op}}$, where

$$\left[\nabla_p \|S_N(0)\|_{\text{op}}\right]_n = \Delta \|S_N(0)\|_{\text{op}} (\psi_n). \tag{24}$$

Since $[\nabla_p \|S_N(0)\|_{\text{op}}]_n$ depends only on ψ_n, parallel computation can be implemented to speed up its computation.

Remark 3: When problem (P) degenerates to the case of stationary actuators, the mobility cost $J_{\text{m}}(\xi,p) = 0$ and the (initial) location ξ_0 is the optimization variable. Such a problem has been studied in [2]. A subgradient-based method is applied to solve the problem after a convex reformulation [20].

A. Numerical example

We use the following values in a numerical example with one mobile actuator:

$$\begin{gathered} \Omega = [0,1], N = 14, m = 1, t_f = 1, \gamma = 0.1, q = 0, q_f = 0, \\ \alpha = 0, \beta = 1, U = \mathbb{R}, P = \mathbb{R}, R = 0.1, a = 0.01, b_1 = 1, \\ \mathcal{Q}(x,y) = \chi(x,y), \mathcal{Q}_f(x,y) = \chi(x,y), \\ K_1(\xi_1(t))(x) = \begin{cases} 100 & |x-\xi(t)| \le \frac{1}{200} \\ 0 & \text{otherwise.} \end{cases} \end{gathered}$$

where $q(\cdot,\cdot)$ and $q_f(\cdot,\cdot)$ are the integral kernels of Hilbert-Schmidt operators $\mathcal{Q}$ and $\mathcal{Q}_f$, respectively, and the indicator function $\chi(x,y) = 1$ if $x = y$ and $\chi(x,y) = 0$ otherwise. Notice that we set the terminal cost in $J_{\text{m}}(\xi,p)$ to 0 by setting $q_f = 0$. The forward propagation of ξ and backward propagation of S_N and Λ_N are computed using the Runge-Kutta method. We use gradient descent with fixed-size step length. The iteration terminates if $|f_{k+1} - f_k| \le 10^{-6}(1 + |f_k|)$, where f_k is the value of the cost function of (AP) at iteration k.

Fig. 1 shows the optimal cost of (AP) when the initial location ξ_0 varies from 0.1 to 0.9. Two cases are considered: $a_{\text{IC}}^2 = 1$ and $a_{\text{IC}}^2 = 10$. As can be seen in Fig. 1, the optimal cost of (AP) is left-right symmetric about the middle of the spatial domain. For each ξ_0, we take the worst-case initial condition $Z_{0,N}$ associated with the maximum eigenvalue of $S_N(0)$ of the optimal solution of (AP) such that

$$\begin{gathered} a_{\text{IC}}^2 \|S_N(0)\|_{\text{op}} = \langle Z_{0,N}, S_N(0) Z_{0,N} \rangle, \\ Z_{0,N}(x) = a_{\text{IC}} v^T \Phi_N(x), \end{gathered}$$

where $v \in \mathbb{R}^N$ is the eigenvector associated with the maximum eigenvalue of $S_N(0)$, Φ_N is a N-dimensional vector function, and $[\Phi_N]_k = \phi_k$. Subsequently, $Z_{0,N}$ and ξ_0 are fed to problem (P1), which is solved using the method in [10]. The optimal cost of (P1) is also shown in Fig. 1, which validates the suboptimality of problem (P)'s solution for (P1): $V^*_{\text{(P)}} \approx V^*_{\text{(AP)}} \ge V^*_{\text{(P1)}}$. Left-right symmetry of $Z_{0,N}$ associated with symmetric initial locations can be observed (not shown), for example, $Z_{0,N}^{0.1}(x) = Z_{0,N}^{0.9}(1-x)$ for $x \in \Omega$ (superscript indicates the value of ξ_0). The symmetry is expected because the diffusion modelled by (1) is isotropic.

Fig. 2 compares the norm of the PDE state and the optimal actuator trajectory solved from (AP) and (P1), respectively, with a_{IC}^2 being 1 or 10 and $\xi_0 = 0.5$. The optimal control of (P1) initially reduces the norm of the PDE state, though the norm at the terminal time is slightly bigger than that associated with (AP). The optimal trajectory of (AP) traverses more area in the spatial domain than that of (P1) since the former is optimal for the worst case, whereas the latter is not. For both (AP) and (P1), the optimal trajectory associated with $a_{\text{IC}}^2 = 10$ is more aggressive than the one associated with $a_{\text{IC}}^2 = 1$ because the former requires the actuator to move more agilely and to dispense more actuation than the latter in order to bring the diffusion to a zero state. The detailed evolution of the PDE state $\mathcal{Z}$ under the control and guidance solved from (AP) with $\xi_0 = 0.5$ is shown in Fig. 3.

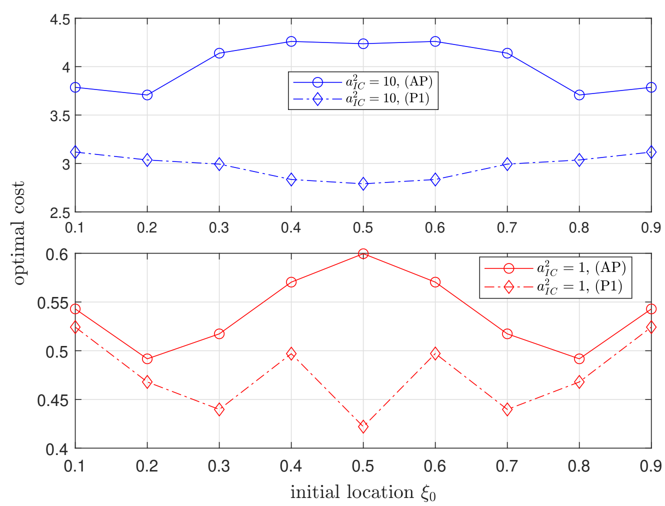

Fig. 1: Optimal costs of (AP) and (P1). The vertical axis of the bottom figure has been scaled up to show the gap between optimal costs of (AP) and (P1).

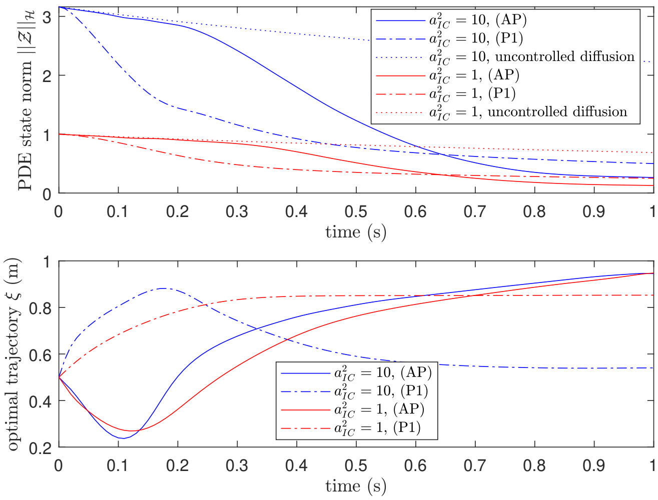

Fig. 2: Norm of the PDE state and the optimal trajectory with $\xi_0 = 0.5$.

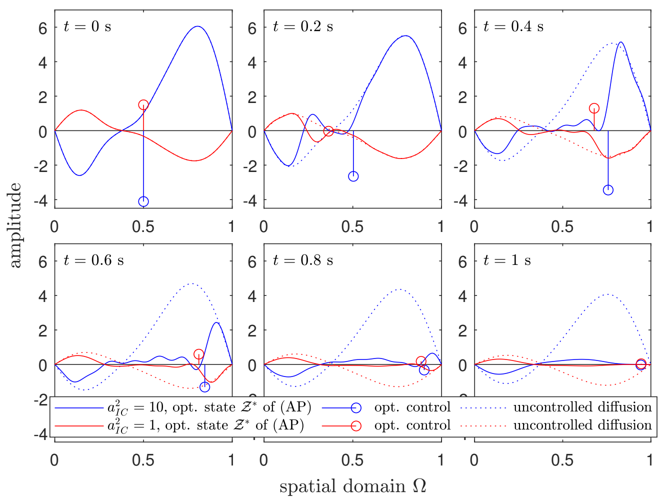

Fig. 3: Evolution of the optimal PDE state $\mathcal{Z}^*$ with optimal control and guidance solved from (AP). The x-coordinate of optimal control indicates the location of the actuator, whereas the y-coordinate indicates its amplitude.

V. Conclusion

This paper formulates an optimization problem for mobile actuators to find the guidance that minimizes the quadratic PDE cost under the worst initial condition, plus a mobility cost. We claim that the solution of the problem studied here is suboptimal for a previously studied problem that has the same cost criterion but with a given initial condition. The suboptimality holds even when the previous problem is subject to the worst initial condition. The suboptimality allows the optimal guidance of the proposed problem to steer the mobile actuators as an initial attempt when the initial condition of the PDE is unknown except for its norm. We turn the proposed problem into an unconstrained optimization and solve it using a gradient-based method. A numerical example demonstrates the suboptimality.

Ongoing work includes the extension of the proposed problem to a 2D domain with advection terms in addition to diffusion. Also, conditions under which the approximated solution converges to the original problem's solution will be investigated.

References

[1] A. Smyshlyaev and M. Krstic, *Adaptive control of parabolic PDEs*. Princeton University Press, 2010.

[2] K. Morris, "Linear-quadratic optimal actuator location," *IEEE Trans. Automatic Control*, vol. 56, no. 1, pp. 113–124, 2010.

[3] X. Wu, B. Jacob, and H. Elbern, "Optimal control and observation locations for time-varying systems on a finite-time horizon," *SIAM J. Control and Optimization*, vol. 54, no. 1, pp. 291–316, 2016.

[4] K. Morris, M. A. Demetriou, and S. D. Yang, "Using H_2-control performance metrics for the optimal actuator location of distributed parameter systems," *IEEE Trans. Automatic Control*, vol. 60, no. 2, pp. 450–462, 2015.

[5] M. A. Demetriou, "Using modified Centroidal Voronoi Tessellations in kernel partitioning for optimal actuator and sensor selection of parabolic PDEs with static output feedback," in *Proc. 56th IEEE Conf. Decision and Control*, 2017, pp. 3119–3124.

[6] K. Morris and S. Yang, "Comparison of actuator placement criteria for control of structures," *J. Sound and Vibration*, vol. 353, pp. 1–18, 2015.

[7] ——, "A study of optimal actuator placement for control of diffusion," in *Proc. 2016 American Control Conf.*, pp. 2566–2571.

[8] D. Xue and N. H. El-Farra, "An optimization-based approach for dynamic actuator scheduling in distributed processes controlled over networks," in *Proc. 2018 American Control Conf.*, pp. 1720–1725.

[9] S. Dubljevic, M. Kobilarov, and J. Ng, "Discrete mechanics optimal control (DMOC) and model predictive control (MPC) synthesis for reaction-diffusion process system with moving actuator," in *Proc. 2010 American Control Conf.*, pp. 5694–5701.

[10] S. Cheng and D. A. Paley, "Optimal control of a 1D diffusion process with a team of mobile actuators under jointly optimal guidance," in *Proc. 2020 American Control Conference*, pp. 3449–3454.

[11] M. A. Demetriou, "Adaptive control of 2-D PDEs using mobile collocated actuator/sensor pairs with augmented vehicle dynamics," *IEEE Trans. Automatic Control*, vol. 57, no. 12, pp. 2979–2993, 2012.

[12] ——, "Guidance of mobile actuator-plus-sensor networks for improved control and estimation of distributed parameter systems," *IEEE Trans. Automatic Control*, vol. 55, no. 7, pp. 1570–1584, 2010.

[13] Y. Chen, Z. Wang, and J. Liang, "Actuation scheduling in mobile actuator networks for spatial-temporal feedback control of a diffusion process with dynamic obstacle avoidance," in *Proc. 2005 IEEE Int. Conf. Mechatronics and Automation*, vol. 2, pp. 752–757.

[14] Y. Chen, Z. Wang, and K. L. Moore, "Optimal spraying control of a diffusion process using mobile actuator networks with fractional potential field based dynamic obstacle avoidance," in *Proc. 2006 IEEE Int. Conf. Networking, Sensing and Control*, pp. 107–112.

[15] R. F. Curtain and H. Zwart, *An introduction to infinite-dimensional linear systems theory*. Springer Science & Business Media, 2012, vol. 21.

[16] S. Cheng and D. A. Paley, "Optimal guidance and estimation of a 1D diffusion process by a team of mobile sensors," *Proc. IEEE 59th Conf. Decision and Control*, pp. 1222–1228, 2020.

[17] J. A. Burns and C. N. Rautenberg, "Solutions and approximations to the Riccati integral equation with values in a space of compact operators," *SIAM J. Control and Optimization*, vol. 53, no. 5, pp. 2846–2877, 2015.

[18] S. Boyd, S. P. Boyd, and L. Vandenberghe, *Convex optimization*. Cambridge University Press, 2004.

[19] J. A. Burns and C. N. Rautenberg, "The infinite-dimensional optimal filtering problem with mobile and stationary sensor networks," *Numerical Functional Analysis and Optimization*, vol. 36, no. 2, pp. 181–224, 2015.

[20] N. Darivandi, K. Morris, and A. Khajepour, "An algorithm for LQ optimal actuator location," *Smart materials and structures*, vol. 22, no. 3, p. 035001, 2013.