Non-Gaussian estimation of a two-vortex flow using a Lagrangian sensor guided by output feedback control

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Abstract—This paper considers the closed-loop navigation of a hypothetical ocean-sampling vehicle in the presence of an idealized ocean-eddy pair. This problem embodies many of the challenges of spatiotemporal ocean sampling using Lagrangian and minimally actuated platforms in ocean currents. We extend an existing guidance strategy known as the Boundary-Touring Algorithm (BTA) to steer a self-propelled vehicle to a unique streamline in a two-vortex flow. The Gaussian Mixture Kalman Filter (GMKF) provides non-Gaussian state estimation of the vortex parameters based on linear observations of Lagrangian sensor position. Taken together, BTA and GMKF constitute a novel guidance framework for adaptive, Lagrangian data assimilation. Results from numerical experiments are presented for a drifting vehicle, a controlled vehicle that has knowledge of the flow field, and a controlled vehicle that navigates based on its own flow field estimate using the BTA/GMKF framework.

I. INTRODUCTION

The problem of spatiotemporal sampling for environmental monitoring contains unique challenges in the oceanographic setting. Ocean-sampling platforms must cover large distances and endure extended deployments. For example, the Argo system, which is a state-of-the-art ocean-observing network, consists of floats that make vertical-profile dives, but otherwise drift passively with the flow [1]. The next ocean-observing network may contain platforms such as gliders, which are vehicles capable of steering and buoyancy-driven propulsion, or other self-propelled, long-endurance vehicles [2],[3]. Utilizing such platforms and their Lagrangian data (i.e., measurements of vehicle position), autonomous control algorithms may exploit flow-field forecasts by using underlying currents for transport and uncertainty reduction.

Previous work [4] identified the importance of coherent structures in the flow field for understanding spatial transport and sampling coverage. Works in the field of Lagrangian data assimilation have optimized the launch site of passively drifting vehicles (e.g., see [5]). Other works have considered energy-optimal [6] and time-optimal [7] paths for continuously self-propelled vehicles. However, a comprehensive framework for adaptive data-driven estimation of a flow field by a hypothetical vehicle with intermittent actuation does not yet exist. Intermittent actuation extends endurance and permits targeted sampling of desirable regions.

This paper focuses on the development of an autonomous estimation and control framework to enable a sampling platform capable of steering and flow-relative propulsion to estimate a potential flow field with unknown parameters. The potential-flow model selected for testing the framework is a two-point-vortex system. The periodic motion of two point vortices in relative equilibrium (e.g., rotating together at a constant rate) represents an idealized model of a naturally-occurring ocean eddy pair. Further, it is a demonstrative problem for studying autonomous navigation, because when viewed from a co-rotating frame this system contains invariant sets that can be used to study the role of coherent structures in navigation and flow-field estimation.

One key component in the control framework is an observability-based navigation algorithm, which relies on the prior finding [3] that boundaries of invariant sets in a divergence-free flow are highly observable due to the eventual distinction of neighboring trajectories by an upstream saddle point. We review an algorithm known as the Boundary-Touring Algorithm (BTA) previously proposed by the authors for touring invariant set boundaries, which yields high observability of the flow field parameters [3]. We extend the BTA by modeling the vehicle as a self-propelled particle in a flow and develop a steering control law that drives it to a unique, closed streamline. The steering control law represents a novel application of an existing transformation to the vehicle speed and heading relative to the flow [8] and an existing flow-free steering algorithm [9] to a time-invariant flow field that is well-described by a streamfunction. We ensure that the vehicle drives to a unique streamline by implementing a steering control built around a Bertrand family of curves extended from the streamline. The streamline steering control uses a virtual cylinder to guide the sampling platform around saddle points as it navigates the boundaries of invariant sets in the two-vortex system.

Another key component in the control framework is a non-Gaussian state estimator—the Gaussian Mixture Kalman Filter (GMKF)—that accommodates nonlinear dynamics and non-Gaussian probability densities by approximating them with a mixture of Gaussians selected to minimize the Bayesian Information Criterion, thereby yielding the simplest (based on the number of parameters) fit of a Gaussian mixture to the data [10]. A square-root Kalman gain [11] is used in place of the traditional Kalman gain to avoid introducing additional sampling error. The BTA and GMKF are

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combined in a novel, guided-Lagrangian data-assimilation framework for estimation of an unknown flow field using feedback of vehicle position measurements. The BTA drives the vehicle along a trajectory of high observability using the best estimate of the flow-field parameters. The GMKF assimilates Lagrangian data (after accounting for the vehicle’s own control effort) to produce more informed estimates of the parameters, thus yielding a new flow-field map for the BTA to tour.

The contributions of this work are as follows: (1) a steering control for the Boundary-Touring Algorithm that drives a self-propelled vehicle to a unique, closed streamline of the two-vortex flow field; (2) implementation of a Gaussian Mixture Kalman Filter (GMKF) with a square-root Kalman gain to estimate the parameters of the two-vortex system using Lagrangian position measurements; and (3) an output-feedback control framework for estimating the two-vortex flow parameters using the BTA and the GMKF. These contributions are significant because the streamline steering controller may be applicable to other time-invariant flows with simple, closed, and regular streamlines. The BTA and GMKF framework for guided-Lagrangian data assimilation may also be used for estimation of other time-invariant flows.

The following provides an outline of the paper. Section II reviews the two-vortex system and observability-based path planning. Section III reviews the Boundary-Touring Algorithm and presents a steering controller that ensures the vehicle drives to a unique streamline of the flow. Section IV describes the nonlinear observer used in the closed-loop navigation strategy. Section V provides simulation results of a vehicle estimating the two-vortex flow by navigating via feedback control. Section VI summarizes the paper and discusses future work.

II. OBSERVABILITY-BASED PATH PLANNING IN THE TWO-VORTEX FLOW

Point vortices from potential flow theory advect under the influence of nearby vortices, but their motion does not include their own contributions to the flow field. Let \( z_j \) denote the location of vortex \( j \) for \( j = 1, 2 \). A nearby vortex \( k \) therefore induces motion on vortex \( j \), yielding [12]

\[
\dot{z}_j = \frac{i \Gamma_k}{2\pi} \frac{z_j - z_k}{|z_j - z_k|^2} \quad \text{for} \quad k \neq j,
\]

where \( \Gamma_k \) is the circulation strength of vortex \( k \) that determines the relative magnitude of the flow induced about \( z_k \). The transformation \( z = \xi e^{(\omega t + \phi)} + z_0 \) changes the reference frame to a co-rotating frame expressed in \( \xi \) coordinates [3]. In this frame, the flow is time-invariant and admits a time-invariant streamfunction [3]

\[
\psi_R(\xi, \overline{\xi}) = -\frac{1}{2\pi} (\Gamma_1 \log|\xi - \xi_1| + \Gamma_2 \log|\xi - \xi_2|) + \frac{\omega}{2} |\xi|^2. \tag{1}
\]

Iso-contours of the streamfunction are streamlines, i.e., lines always tangent to the flow velocity. Therefore, the flow velocity may be written in terms of the streamfunction as

\[
f_R = -2i \frac{\partial \psi_R}{\partial \overline{\xi}}, \tag{2}
\]

where the \( (\cdot)_R \) subscript denotes the co-rotating frame. A drifting vehicle located at \( \xi \) and advected with the flow therefore has dynamics \( \dot{\xi} = -2i \frac{\partial \psi_R}{\partial \overline{\xi}} \). The flow field of the two-vortex system in the co-rotating frame is shown in Fig. 1. Separating boundaries or separatrices (shown in black) are the stable and unstable manifolds of the saddle points. These boundaries form six invariant sets (regions from which a vehicle cannot escape without exerting control). The saddle points in the two-equal vortex system are located at \( \xi = \{0, \pm \sqrt{2}d\} \), where \( d = |\xi_1 - \xi_2| \) is the vortex separation. Locating saddle points in the flow is an important first step in the identification of the invariant-set boundaries. This process is performed adaptively in a control loop for a vehicle navigating using the estimated flow in Section V.

Observability of a linear system describes one’s ability to infer the initial state of the system by observing its output. Krner and Ide [13] presented the empirical observability Gramian \( W_o(t_0, t_1) \) for nonlinear systems, which is constructed by considering the sensitivity of the system response to perturbations in the initial state or system parameter values. The empirical observability Gramian is given component-wise by [13],[3]

\[
W_o(k, j) = \frac{1}{4\epsilon_k \epsilon_j} \int_{t_0}^{t_1} [Y^{+k}(\tau) - Y^{-k}(\tau)]^T \times [Y^{+j}(\tau) - Y^{-j}(\tau)] \, d\tau, \quad \text{for} \quad k, j = 1, \ldots, n,
\]

where \( Y^{\pm k}(t) = X(t) \pm \epsilon_k e_k \) is the system output in response to a perturbed initial state \( X^{\pm k}(t_0) = X(t_0) \pm \epsilon_k e_k \), which has been perturbed along the direction of the \( e_k \) unit vector [3]. For the two-vortex problem with equal strength vortices, the state

\[1\text{The complex partial derivative operators are } \frac{\partial}{\partial \xi} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \text{ and } \frac{\partial}{\partial \overline{\xi}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).\]
Fig. 2: (a) Example of a vehicle performing a boundary tour, using tour \( \{1, 2, 4, 5, 4, 2, 1\} \) [3]. (b) A vehicle navigating to a uniquely specified streamline is shown in blue. Two streamlines with identical streamfunction values are shown in magenta. A vehicle using the controller from [3] to target a non-unique streamfunction value is shown in green.

vector is

\[
X = (\Gamma, \text{Re}(z_1), \text{Im}(z_1), \text{Re}(z_2), \text{Im}(z_2), \text{Re}(z), \text{Im}(z))^T, \tag{4}
\]

where \( z \) denotes the location of a drifting or guided Lagrangian sensor. The reciprocal of the smallest eigenvalue of \( W_\kappa (t_0, t_1) \) is called the unobservability index and represents a measure of how unobservable the least observable mode of the system is [13]. (Hence, a small unobservability index corresponds to high observability.)

Prior work [3] analyzed the observability of orbits in the two-vortex system with the aim of planning a path for a vehicle that spends most of its time drifting with the flow, actuating infrequently to switch streamlines. Orbits approaching the invariant-set boundaries in the two-vortex system were shown to possess the greatest empirical observability of the flow field parameters given position measurements from candidate drifter trajectories, as shown in Fig. 1(b). Using knowledge of the underlying flow field, i.e., a flow-field map, the authors constructed the Boundary-Touring Algorithm (BTA), in which the sampling vehicle visits boundaries of the invariant sets using intermittent actuation. The two-part algorithm consists of a streamline controller that drives the vehicle to a desired streamfunction value and a saddle-navigation controller that makes use of a virtual cylinder to plan vehicle paths that circumvent saddle points. Saddle-point avoidance is particularly important in an estimated flow field where widely diverging vehicle trajectories may result from approaching it. A vehicle executing the BTA navigates around the boundaries of a user-designed tour of invariant sets (i.e., an ordered list of adjacent regions, usually beginning and ending in the same region) to observe the flow. Fig. 2(a) shows a vehicle navigating a known flow field according to the BTA [3]. Actuation is applied near saddle locations, where multiple regions of the flow meet. The vehicle navigates around a virtual cylinder to reach the regions specified in the tour.

III. STREAMLINE CONTROL AND THE BOUNDARY-TOURING ALGORITHM

This work utilizes the BTA [3] to navigate known and estimated flow fields (see Section V). The steering controller detailed below drives the sampling vehicle to a unique streamline in the flow, as shown in Fig. 2(b). The steering controller is based on a self-propelled particle model with second-order dynamics. Since the vehicle travels along streamlines, it may shut off its propulsion and drift after converging to the desired path. The controller combines an existing flow-relative transformation [8] with an existing flow-free control using a Bertrand family of curves around a closed streamline of the flow.

A self-propelled particle model for a vehicle, located at \( z = x + iy \) in the \( \mathbb{C} \) plane, and for which control is applied gyroscopically to the steering rate, is [8], [14]

\[
\dot{z} = se^{i\theta} + f \quad \text{with} \quad \dot{\theta} = u, \tag{5}
\]

where \( f \) represents the flow velocity at \( z \). For a time-invariant flow field well-described by stream function \( \psi(z, \tau) \), we have \( f = -2i\frac{\partial \psi}{\partial x} \) [3].

A transformation of Paley and Peterson [8] re-writes the model (5) as [8]

\[
\dot{z} = \alpha e^{i\beta} \quad \text{with} \quad \dot{\beta} = \nu, \tag{6}
\]

using the flow-relative vehicle speed \( \alpha = |se^{i\theta} + f| \), the flow-relative velocity orientation \( \beta = \arg(se^{i\theta} + f) \), and the flow-relative control input \( \nu \) [8]. The control inputs between the two models are related by [8]

\[
u = \frac{\nu - \langle f, e^{i\beta} \rangle}{1 - \alpha^{-1} \langle f, e^{i\beta} \rangle}, \tag{7}
\]

where \( \langle f, e^{i\beta} \rangle = \frac{\partial f}{\partial z} \dot{z} + \frac{\partial f}{\partial \bar{z}} \bar{z} \). Note that (7) has a singularity if the vehicle is unable to make forward progress, that is if \( \langle f, e^{i\beta} \rangle = \alpha \), or equivalently if \( \langle f, e^{i\beta} \rangle = -s \) [8]. In strong flows for which the vehicle may not always be able to make forward progress, a saturation function may be applied to \( u \) to handle large control excursions [14].

Zhang and Leonard [9] use model (6) to derive a curve-tracking controller to follow a simple, regular, finite closed curve \( \gamma_0(\cdot) \) parameterized by arc-length \( s \). Converging to \( \gamma_0 \) is accomplished by construction of a scalar orbit function \( \Phi(z) \) for which \( \gamma_0 \) is a level curve (i.e., \( \Phi(\gamma_0(\cdot)) \) is constant along the arc-length) and for which certain technical requirements are satisfied [9]. The vehicle utilizes the relative angle \( \eta \) between its own path frame (Frenet-Serret frame) and the path frame of a particle located at \( z \) and traveling along a level-curve of \( \Phi \) to steer asymptotically to a desired level curve of \( \Phi \) [9]. Let \( \{b_1, b_2\} \) represent the path frame of the vehicle located at \( z \), where \( b_1 = e^{i\beta} \) and \( b_2 = ib_1 \). Let \( \{c_1, c_2\} \) represent the path frame of a virtual vehicle collocated at \( z \) and traveling on a level-curve of \( \Phi \) [9]. We have\(^3\) [9]

\[
\cos \eta = \langle b_1, c_1 \rangle = \langle b_2, c_2 \rangle.
\]

See Fig. 3 for an illustration of these quantities.

\(^2\)We note that the separating boundaries in the two-vortex flow do not meet the regularity condition at saddle points. However, inserting of virtual cylinder into the flow field (using Eqn. (15) in [3]) allows for the numerical construction of boundaries avoiding saddles and meeting these conditions.

\(^3\)The inner product of complex numbers \( a \) and \( b \) is given by \( \langle a, b \rangle = \text{Re}(ab) \), where \( \overline{\cdot} \) denotes complex conjugation.
If curve γ₀ is a member of a parameterized family of curves γₜ(·; λ), such as a family of concentric ellipses, then the orbit function Φ may be constructed using the scalar parameter λ [9]. If γ₀ is a more general (simple, regular, and closed) curve, then an orbit function may be constructed using a Bertrand family of curves [9], i.e.,

\[ γₜ(s) = γ₀(s) + λc₂(s), \]

in which additional family members are formed by offsetting from γ₀ by a distance of |λ| perpendicular to the curve (in either the positive or negative c₂ direction, depending on the sign of λ) [9]. The orbit function may be defined to be \( Φ(z) = λ \) if z lies on the curve γ₀ [9]. The arc-length s is measured along the reference orbit [9]. (The resulting control law \( ν \) that drives to γ₀ is given in the Appendix.)

A unique orbit of the flow (using the Fundamental Theorem of Calculus) is

\[ γ₀(t) = z₀ + \int_{0}^{t} -2ν\frac{∂Φ}{∂z} dt, \quad 0 < t \leq T, \]

where \( z₀ \) is a point lying on the orbit and T is the period of the orbit. To steer to a unique orbit of the flow, we construct a Bertrand family of curves γₜ around the reference orbit and define the orbit function Φ(z) as described previously. By the construction of a Bertrand family, the direction c₂ is always perpendicular to the curve, so it also lies along or in the direction opposite to the gradient of the orbit function. Following [21], we choose the convention

\[ 2\frac{∂Φ}{∂z} = c₂. \]

The necessary derivatives of the orbit function to implement \( ν \) given in (14) may be found in terms of the streamfunction as

\[ \frac{∂^2Φ}{∂z∂z} = \frac{1}{2|\frac{∂ψ}{∂z}|} \begin{pmatrix} \frac{∂^2ψ}{∂z^2} \end{pmatrix} c₁ \]

\[ \frac{∂^2Φ}{∂z^2} = \frac{1}{2|\frac{∂ψ}{∂z}|} \begin{pmatrix} \frac{∂^2ψ}{∂z^2} \end{pmatrix} c₁. \]

The right-hand sides of these equations are evaluated at the point on the reference orbit nearest to the location of the vehicle at z.

IV. GAUSSIAN MIXTURE KALMAN FILTER WITH SQUARE-ROOT GAIN

We use a Gaussian Mixture Kalman Filter (GMKF) for non-Gaussian estimation because it performs nonlinear forecasts of state uncertainty and is capable of handling the nonlinear dynamics present in Lagrangian data assimilation. Gaussian mixture-based filters have previously appeared in literature in a variety of forms (see, e.g., [15],[16],[17],[10], and [18]); This section is based primarily on the filter of Sonderegger and Lerumasiaux [10], known as the GMMD-DO filter because it combines Gaussian mixture models and dynamically orthogonal field equations. The GMMD-DO filter differs from other mixture filters in the automated selection of the number of Gaussians in the mixture. The GMMD algorithm we present differs from GMM-DO in its use of a square-root Kalman gain. We also directly forecast state realizations and do not use the DO framework.

Let \( w_j, \ j = 1,\ldots, M \), be scalar weights such that \( \sum_{j=1}^{M} w_j = 1 \). Let \( X_j \) and \( P_j \) be the mean vector and covariance matrix respectively for a multivariate Gaussian \( \mathcal{N}(X; X_j, P_j) \), \( j = 1,\ldots, M \). The weighted sum of the M Gaussian densities [10]

\[ p(X; \{(w_j, X_j, P_j)\}_{j=1}^{M}) = \sum_{j=1}^{M} w_j \mathcal{N}(X; X_j, P_j) \]

is a valid probability density function (pdf) known as a Gaussian mixture that integrates to unity and has an analytical representation. Through the selection of the weights, means, covariances, and number of mixture components, highly non-Gaussian distributions may be represented.

Traditional ensemble/particle-based methods represent a pdf using a sparse support of ensemble members (i.e., a Monte Carlo sampling of realizations) [19]. This representation enables nonlinear propagation of the uncertainty in the forecast step of the filter. Unfortunately, many particle filters suffer from degeneracy issues due to the sparsity of the pdf representation [19]. Kernel-based approaches address this issue by periodically creating a full density estimate from the ensemble sample so that the state space is fully supported and resampling may be performed [19]. Unfortunately, such approaches invariably require the arbitrary choice of fitting parameters such as the kernel bandwidth [10]. For Gaussian mixtures, given a specific choice for mixture complexity, an Expectation-Maximization algorithm may be applied to select automatically the weights, means, and covariances of the Gaussians to best fit the ensemble [20]. A key contribution of [10] is the use of the Bayesian Information Criterion (BIC) for the automatic selection of the mixture complexity as well. The BIC may be (approximately) expressed as [10]

\[ \text{BIC} = -2 \sum_{j=1}^{N} \log p(X_j|Ω_{ML}; M) + K_M \log N, \]

where \( K_M \) is the number of parameters in the model, \( Ω_{ML} \) is the maximum likelihood set of parameters (produced by the EM algorithm), and \( N \) is the number of ensemble members. For a multivariate Gaussian mixture, \( K_M = M \left(2n + \frac{n(n-1)}{2} + 1\right) \) is the number of free parameters, where \( n \) is the dimension of the state vector. Note that the BIC has two components: the first component evaluates the goodness-of-fit for the model of complexity M and the second component is a penalty on the overall model complexity [10]. By sequentially evaluating models of increasing
TABLE I: Gaussian Mixture Kalman Filter [10], modified for a square-root Kalman gain [11].

<table>
<thead>
<tr>
<th>Input: GMM of prior pdf</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameters: N, maxComplexity, and covariance matrices C, R</td>
</tr>
<tr>
<td>Output: GMM of analysis pdf</td>
</tr>
</tbody>
</table>

1. Sample $N$ particles from the prior pdf.
2. Integrate the particles forward in time with process noise sampled from $N(0, C)$.
3. Fit a minimal GMM to the forecast ensemble
   - EM algorithm to fit GMM with 1 Gaussian.
   - Evaluate BIC.
   - for $m = 2$ to maxComplexity do
     - EM algorithm to fit GMM with $m$ Gaussians.
     - Evaluate BIC.
   - If BIC increases, select previous GMM, and break loop.
4. Assimilate measurements
   - Calculate the analysis weight for each Gaussian
     $$ w_j^a = \frac{w_j^f N(Y; H\mu_j^f, HP_j^f HT + R)}{\sum_{q=1}^M N(Y; H\mu_q^f, HP_q^f HT + R)} $$
   - Calculate the square-root Kalman gain for each Gaussian
     $$ B_j = \left( HP_j^f HT + R \right)^{1/2} $$
     $$ K_j = P_j^f HT B_j^{-T} \left( B_j + R^{1/2} \right)^{-1} $$
   - Calculate the analysis mean for each Gaussian
     $$ \mu_j^a = \mu_j^f + K_j(Y - H\mu_j^f) $$
   - Calculate the analysis covariance for each Gaussian
     $$ P_j^a = (I - K_jH)P_j^f $$

For Lagrangian data-assimilation applications, the observation-operator $H$ linearly extracts the vehicle position from the state vector. That is,

$$ Y = HX + \mu \quad \text{with} \quad \mu \sim N(0, R), $$

where $\mu$ is zero-mean measurement noise with covariance $R$. (The state vector is given by (4).) In the case of a (single) Gaussian forecast pdf, Gaussian measurement noise, and a linear observation operator, the Kalman-analysis equations represent the optimal approach to Bayesian assimilation of a measurement. In the case of a mixture of Gaussians, the Kalman-analysis equations may be augmented with a weight-analysis equation to yield the proper application of Bayes theorem for each component in the mixture [10]. Table I presents these equations and the overall GMKF algorithm.

For $M=1$, the GMKF reduces to an Ensemble Kalman Filter in which only a single Gaussian is used to represent the prior (forecast) and posterior (analysis) densities. For $M > 1$, the GMKF may be viewed as a collection of Ensemble Kalman Filters operating in parallel [10]. It has been well-established that Ensemble Kalman Filters may systematically underestimate the posterior covariance if all ensemble members are updated with the same observation [11]. Two approaches have addressed this issue [11]: (1) using perturbed observations in which each ensemble member assimilates a different realization of the measurement; and (2) the use of a square-root Kalman gain that obviates the need for perturbed observations. We use a square-root Kalman gain here because it has been shown to avoid the introduction of additional sampling error that occurs in the method of perturbed observations [11].

Fig. 4 shows the functioning of the GMKF on an estimation task for the two-vortex system. (These data are from the experiment in Section V, which uses a guided-Lagrangian sensor with a known flow.) Fig. 4(a) shows the number of Gaussians automatically selected by the filter to represent the forecast pdf. Fig. 4(b) depicts the marginal posterior pdf for the location of a vortex (at an early time in the experiment). The mixture model enables the filter to represent the non-Gaussian uncertainty necessary to track the true vortex location (shown as a red circle). The center of vorticity is a red plus; the GMKF estimate is a black circle.

V. CLOSED-LOOP CONTROL USING THE FLOW FIELD ESTIMATE

This section combines the BTA and GMKF to construct a novel, guided-Lagrangian data-assimilation framework for estimation of the two-vortex flow. A vehicle in this framework executes the BTA to traverse trajectories expected to yield high observability of the flow-field parameters. The vehicle applies the GMKF to estimate the flow-field parameters using position measurements. When the flow-field parameters are not known a priori, the vehicle implements output feedback control to adapt the flow map according to the measurements.

We present three numerical experiments of guided-Lagrangian sampling of the two-vortex flow. One vehicle uses a prescribed navigation controller (i.e., the controller knows the true flow, but the estimator does not) and another uses an output feedback controller based on the vehicle’s estimate of the flow field (i.e., neither the controller nor the estimator know the true flow). Both vehicles are deployed from the same inertial location. The third vehicle is a drifter launched from the same location as the first two.

Fig. 5 shows trajectories of the three vehicles; the path of the drifting vehicle (confined to Region 2) is shown...
Fig. 5: Autonomous navigation of the two-vortex system using (a)–(b) a known flow map, and (c)–(d) output feedback to estimate the flow map. Plots (a) and (c) show paths traversed in the co-rotating frame. The path of a drifting vehicle is shown in magenta. Plots (b) and (d) show paths traversed in the inertial frame. Green markers are position measurements. Red circles indicate vortex locations and hollow black circles show estimates of vortex position.

in magenta. The co-rotating frame in Figs. 5(a) and 5(c) is based on the true vortex-pair rotation rate $\omega = \frac{\Gamma}{\pi d}$ [12]. The separatrices and saddle points are also based on the true parameter values. The vehicles are launched in Region 2 and guided along the boundary tour $(4, 5, 4, 6, 4)$. The tour $(4, 5, 4, 6, 4)$ contains many close approaches to saddle points, and minimizes the time spent at the outer edges of Regions 2 and 3, where the flow speed is reduced.

Although the vehicle with the estimated map meanders from its intended path initially, it eventually converges to the desired tour. Further, the inertial trajectories in Figs. 5(c) and 5(d) yield similar coverage patterns between the two actuated vehicles. Note that the vehicle with the estimated map violates the cylinder stay-out zones since it does not precisely know the true saddle-point locations. Selecting the virtual cylinder radii based on flow field uncertainty is the subject of ongoing work.

Fig. 6 compares the estimation performance of the three vehicles. Fig. 6(a) shows the $L_2$ norm of the state error versus time. Fig. 6(c) shows the percent error in circulation strength. Figs. 6(b) and 6(d) show errors in the center of vorticity $z_{cv}$ and the separation distance $d$, which are quantities conserved by the vortex dynamics. The drifting vehicle correctly estimates $\Gamma$, but its overall performance is poor. The periodic estimate excursions visible in Fig. 6(d) correspond to times when the drifter is far from the center of the vorticity. The guided vehicles yield comparable estimation performance, which is expected since the vehicle with the estimated map converges to the same route as the vehicle with the known map. The vehicle with the estimated map performs better in estimation of $\Gamma$; this may be attributed to additional exploration during the initial transient phase.

VI. CONCLUSION

This paper presents a principled approach to estimate the parameters of a two-vortex flow that models a double-eddy system in the ocean. The estimation and control framework guides a self-propelled Lagrangian sensor along highly observable trajectories, which are the boundaries of invariant sets. The two main components of the framework are the Boundary-Touring Algorithm (BTA) and the Gaussian Mixture Kalman Filter (GMKF). We extend the BTA by presenting a steering controller that combines a flow-relative transformation and steering control built around a Bertrand family of curves to steer to a specified, closed streamline of the two-vortex flow. The GMKF is a dynamic nonlinear observer that produces estimates of the flow field parameters used to adapt the uncertain map of the flow environment. In ongoing work, we are extending this method to other time-invariant and time-varying flows.

VII. APPENDIX

The following proposition summarizes the applicable work of [9] and [21] which has been adapted for complex variables.

**Proposition 1 (Bertrand curve following [9]):** For simple, regular, closed curve $\gamma_0(\cdot)$ and an associated family of Bertrand curves $\gamma_\lambda(\cdot)$, define the orbit function $\Phi(z) = \lambda$ when $z$ lies on the curve $\gamma_\lambda$. Given a scalar function $h$...
meeting the technical requirements listed in [9] and the self-propelled vehicle model (6), the steering control law

\[
\nu = \kappa_a \cos \eta + \kappa_b \sin \eta - 2 \frac{dh(\Phi, \lambda_{des})}{d\Phi} \cos^2 \frac{\eta}{2} + K_1 \sin \frac{\eta}{2}
\]

\[
\kappa_a = -s \left( c_1, 2 \frac{\partial^2 \Phi}{\partial z \partial \sigma} c_1 + 2 \frac{\partial^2 \Phi}{\partial \tau^2} c_1 \right)
\]

\[
\kappa_b = -s \left( c_1, 2 \frac{\partial^2 \Phi}{\partial z \partial \sigma} c_2 + 2 \frac{\partial^2 \Phi}{\partial \tau^2} c_2 \right)
\]  (14)

drives the vehicle asymptotically to the Bertrand curve \( \gamma_{\lambda_{des}}(\cdot) \).

\[ \text{Proof:} \] See [21] for the details of the proof.

VIII. ACKNOWLEDGMENTS

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REFERENCES


